Some history of Latin squares in experiments

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Abstract

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However, there is evidence of their much earlier use in experiments.

They have led to interesting special cases,
arguments, counter-intuitive results,
and a spectacular solution to an old problem.

What is a Latin square?

Definition

Let n be a positive integer.
A Latin square of order n is an $n \times n$ array of cells in which n symbols are placed, one per cell, in such a way that each symbol occurs once in each row and once in each column.

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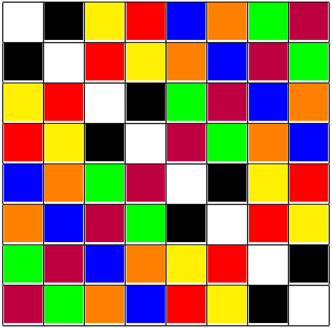
The symbols may be letters, numbers, colours, …

A Latin square of order 8



A Latin square of order 6

E	B	F	A	C	D
B	C	D	E	F	A
A	E	C	B	D	F
F	D	E	C	A	B
D	A	B	F	E	C
C	F	A	D	B	E

A stained glass window in Caius College, Cambridge

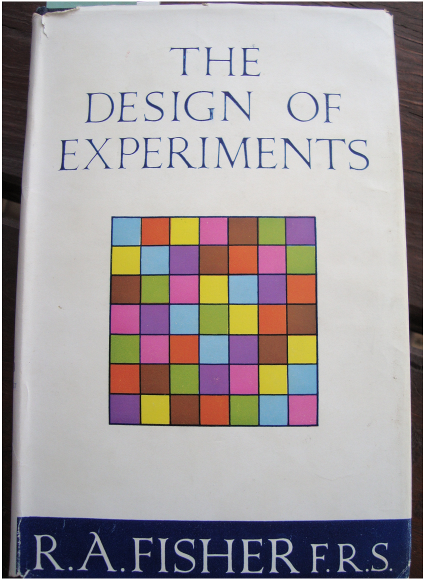
photograph by
J. P. Morgan

And on the opposite side of the hall

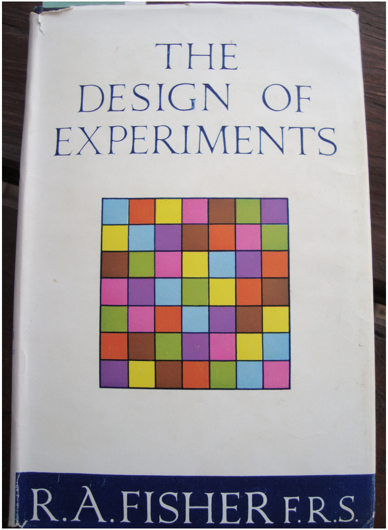
And on the opposite side of the hall

R. A. Fisher promoted the use of Latin squares in experiments while at Rothamsted (1919–1933) and his 1935 book *The Design of Experiments*.

Stained glass window: book cover



A Latin square of order 7



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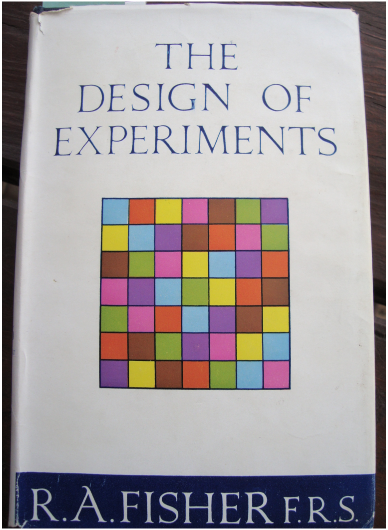

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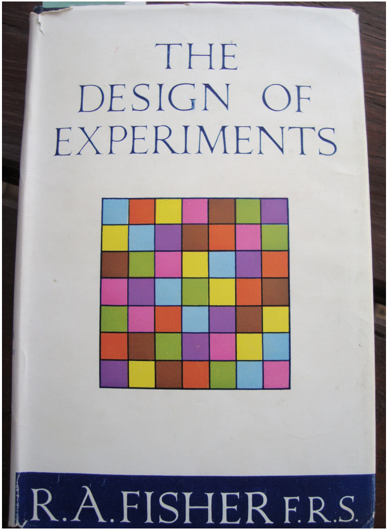

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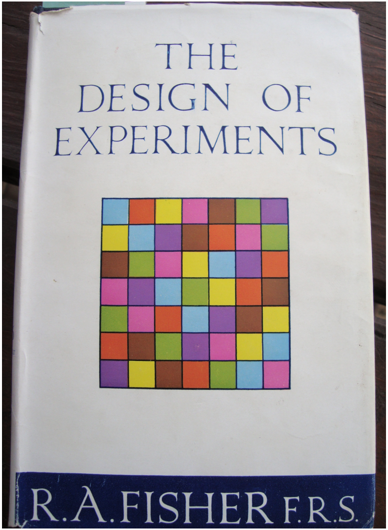

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Why is it called 'Latin'?

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Agricultural field trials, with rows and columns corresponding to actual rows and columns on the ground (possibly the width of rows is different from the width of columns).

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"…on any given field agricultural operations, at least for centuries, have followed one of two directions, which are usually those of the rows and columns; consequently streaks of fertility, weed infestation, etc., do, in fact, occur predominantly in those two directions."

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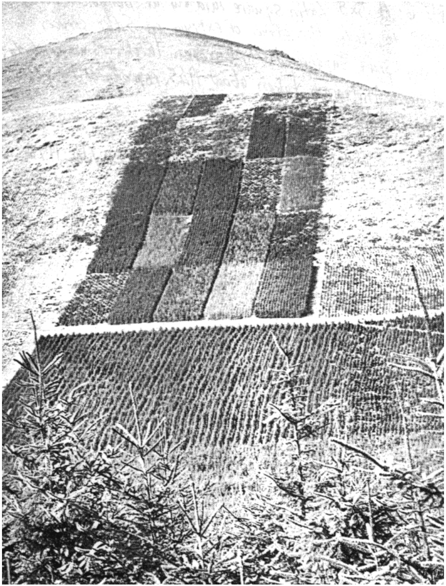
This assumption is dubious for field trials in Australia.

An experiment on potatoes at Ely in 1932

E	B	F	A	C	D
B	C	D	E	F	A
A	E	C	B	D	F
F	D	E	C	A	B
D	A	B	F	E	C
C	F	A	D	B	E

Treatment	A	B	C	D	E	F
Extra nitrogen	0	0	0	1	1	1
Extra phosphate	0	1	2	0	1	2

A forestry experiment



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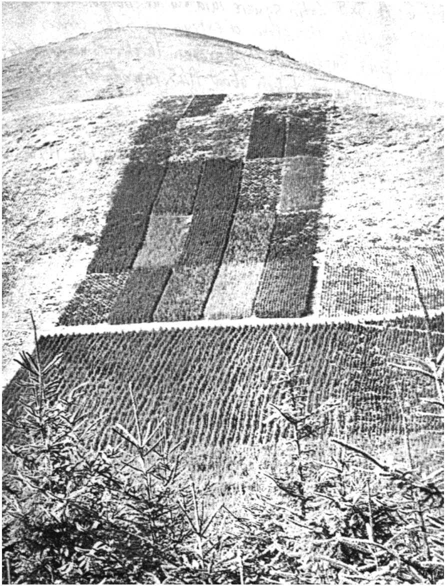

Experiment on a hillside near Beddgelert Forest, designed by Fisher and laid out in 1929

Other sorts of rows and columns: animals

An experiment on 16 sheep carried out by François Catté de Palluel, reported in *Annals of Agriculture* in 1790. They were fattened on the given diet, and slaughtered on the date shown.

slaughter date	Breed			
	Ile de France	Beauce	Champagne	Picardy
20 Feb	potatoes	turnips	beets	oats & peas
20 Mar	turnips	beets	oats & peas	potatoes
20 Apr	beets	oats & peas	potatoes	turnips
20 May	oats & peas	potatoes	turnips	beets

Other sorts of rows and columns: plants in pots

An experiment where treatments can be applied to individual leaves of plants in pots.

	plant			
height	1	2	3	4
1	*A*	*B*	*C*	*D*
2	*B*	*A*	*D*	*C*
3	*C*	*D*	*A*	*B*
4	*D*	*C*	*B*	*A*

Graeco-Latin squares

A	B	C
C	A	B
B	C	A

α	β	γ
β	γ	α
γ	α	β

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When the two Latin squares are superposed,
each Latin letter occurs exactly once with each Greek letter.

$A \quad \alpha$	$B \quad \beta$	$C \quad \gamma$
$C \quad \beta$	$A \quad \gamma$	$B \quad \alpha$
$B \quad \gamma$	$C \quad \alpha$	$A \quad \beta$

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Euler called such a superposition a ‘Graeco-Latin square’.

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Euler called such a superposition a 'Graeco-Latin square'.
The name 'Latin square' seems to be a back-formation from this.

Pairs of orthogonal Latin squares

Definition

A pair of Latin squares of order n are orthogonal to each other if, when they are superposed, each letter of one occurs exactly once with each letter of the other.

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We have just seen a pair of orthogonal Latin squares of order 3.

Mutually orthogonal Latin squares

Definition

A collection of Latin squares of the same order is mutually orthogonal if every pair is orthogonal.

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Example ($n = 4$)

$A\alpha 1$	$B\beta 2$	$C\gamma 3$	$D\delta 4$
$B\gamma 4$	$A\delta 3$	$D\alpha 2$	$C\beta 1$
$C\delta 2$	$D\gamma 1$	$A\beta 4$	$B\alpha 3$
$D\beta 3$	$C\alpha 4$	$B\delta 1$	$A\gamma 2$

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$C\delta 2$	$D\gamma 1$	$A\beta 4$	$B\alpha 3$
$D\beta 3$	$C\alpha 4$	$B\delta 1$	$A\gamma 2$

Theorem

If there exist k mutually orthogonal Latin squares $L_1, \ldots, L_k$ of order n, then $k \leq n - 1$.

When is the maximum achieved?

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If n is a power of a prime number then there exist $n - 1$ mutually orthogonal Latin squares of order n.

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For example, $n = 2, 3, 4, 5, 7, 8, 9, 11, 13, \ldots$.

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R. A. Fisher and F. Yates: *Statistical Tables for Biological, Agricultural and Medical Research*. Edinburgh, Oliver and Boyd, 1938.
This book gives a set of $n-1$ MOLS for $n = 3, 4, 5, 7, 8$ and 9. The set of order 9 is not made by the usual finite-field construction, and it is not known how Fisher and Yates obtained this.

An industrial experiment using MOLS

L. C. H. Tippett: Applications of statistical methods to the control of quality in industrial production. Manchester Statistical Society (1934). (Cited by Fisher, 1935)

A cotton mill has 5 spindles, each made of 4 components. Why is one spindle producing defective weft?

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Period	i	ii	iiii	iv	v
1	$A\alpha 1$	$B\beta 2$	$C\gamma 3$	$D\delta 4$	$E\varepsilon 5$
2	$E\delta 3$	$A\varepsilon 4$	$B\alpha 5$	$C\beta 1$	$D\gamma 2$
3	$D\beta 5$	$E\gamma 1$	$A\delta 2$	$B\varepsilon 3$	$C\alpha 4$
4	$C\varepsilon 2$	$D\alpha 3$	$E\beta 4$	$A\gamma 5$	$B\delta 1$
5	$B\gamma 4$	$C\delta 5$	$D\varepsilon 1$	$E\alpha 2$	$A\beta 3$

1st component	2nd component	3rd component	4th component
i–v	A–E	α–ε	1–5

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3	$D\beta 5$	$E\gamma 1$	$A\delta 2$	$B\varepsilon 3$	$C\alpha 4$
4	$C\varepsilon 2$	$D\alpha 3$	$E\beta 4$	$A\gamma 5$	$B\delta 1$
5	$B\gamma 4$	$C\delta 5$	$D\varepsilon 1$	$E\alpha 2$	$A\beta 3$

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How to randomize? I

R. A. Fisher: The arrangement of field experiments. *Journal of the Ministry of Agriculture*, **33** (1926), 503–513.

> Systematic arrangements in a square … have been used previously for variety trials in, for example, Ireland and Denmark;

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> Systematic arrangements in a square … have been used previously for variety trials in, for example, Ireland and Denmark; but the term "Latin square" should not be applied to any such systematic arrangements. The problem of the Latin Square, from which the name was borrowed, as formulated by Euler, consists in the enumeration of *every possible* arrangement, subject to the conditions that each row and each column shall contain one plot of each variety. Consequently, the term Latin Square should only be applied to a process of randomization by which one is selected at random out of the total number of Latin Squares possible, …

How many different Latin squares of order n are there?

Are these two Latin squares the same?

A	B	C
C	A	B
B	C	A

1	2	3
3	1	2
2	3	1

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To answer this question, we will have to insist that all the Latin squares use the same symbols, such as $1, 2, \ldots, n$.

Reduced Latin squares, and equivalence

Definition

A Latin square is reduced if the symbols in the first row and first column are $1, 2, \ldots, n$ in natural order.

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Definition

Latin squares L and M are equivalent if there is a permutation f of the rows, a permutation g of the columns and permutation h of the symbols such that

$$\begin{array}{llllllll} \text{symbol} & s & \text{is in row} & r & \text{and column} & c & \text{of} & L \\ & & & \Longleftrightarrow & & & & \\ \text{symbol} & h(s) & \text{is in row} & f(r) & \text{and column} & g(c) & \text{of} & M. \end{array}$$

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Theorem
If there are m reduced squares in an equivalence class of Latin squares of order n, then the total number of Latin squares in the equivalence class is $m \times n! \times (n-1)!$.

Order 3

There is only one reduced Latin square of order 3.

1	2	3
2		
3		

Order 3

There is only one reduced Latin square of order 3.

1	2	3
2	3	1
3	1	2

Order 4

There are two equivalence classes of Latin squares of order 4.

1	2	3	4
2	3	4	1
3	4	1	2
4	1	2	3

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

Order 4

There are two equivalence classes of Latin squares of order 4.

1	2	3	4
2	3	4	1
3	4	1	2
4	1	2	3

cyclic

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

non-cylic group

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1	2	3	4
2	3	4	1
3	4	1	2
4	1	2	3

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2	1	4	3
3	4	1	2
4	3	2	1

non-cylic group

more 2×2 Latin subsquares

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3	4	1	2
4	1	2	3

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

cyclic

non-cylic group

more 2×2 Latin subsquares

3 reduced squares

1 reduced square

MacMahon's counting

"… problem of the Latin square. I have given the mathematical solution and you will find it in my *Combinatory Analysis*, Vol. 1, p. 250.

For $n = 2$,	no.	of	arrangements is	2
3,	"	"	"	12
4,	"	"	"	576
5,	"	"	"	149 760

and I have not calculated the numbers any further."

P. A. MacMahon
letter to R. A. Fisher,
30 July 1924
(selected correspondence edited by J. H. Bennett)

Correction

Fisher divided by $n! \times (n-1)!$ to obtain the number of reduced Latin squares, which he pencilled in.

				all	reduced
For $n = 2$,	no.	of	arrangements is	2	1
3,	″	″	″	12	1
4,	″	″	″	576	4
5,	″	″	″	149 760	52

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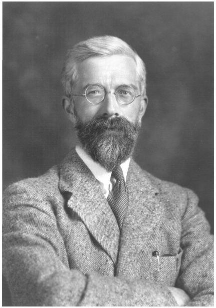

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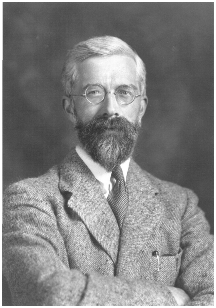

By September 1924 they had agreed that the number of reduced Latin squares of order 5 was 56, not 52.

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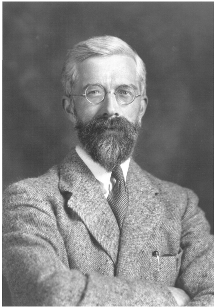

By September 1924 they had agreed that the number of reduced Latin squares of order 5 was 56, not 52.

Euler had already published this result in 1782; and so had Cayley in a 1890 paper called ‘On Latin squares’.

Order 5

There are two equivalence classes of Latin squares of order 5.

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

1	2	3	4	5
2	1	4	5	3
3	4	5	1	2
4	5	2	3	1
5	3	1	2	4

Order 5

There are two equivalence classes of Latin squares of order 5.

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

cyclic

1	2	3	4	5
2	1	4	5	3
3	4	5	1	2
4	5	2	3	1
5	3	1	2	4

not from a group

Order 5

There are two equivalence classes of Latin squares of order 5.

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

cyclic

no 2×2 Latin subsquare

1	2	3	4	5
2	1	4	5	3
3	4	5	1	2
4	5	2	3	1
5	3	1	2	4

not from a group

has a 2×2 Latin subsquare

Order 5

There are two equivalence classes of Latin squares of order 5.

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

cyclic

no 2×2 Latin subsquare

6 reduced squares

1	2	3	4	5
2	1	4	5	3
3	4	5	1	2
4	5	2	3	1
5	3	1	2	4

not from a group

has a 2×2 Latin subsquare

50 reduced squares

So how is the experimenter to obtain a Latin square?

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This includes every reduced Latin square of orders 2, 3, 4 (and 5?), and one Latin square from each equivalence class of Latin squares of order 6.

Numbers of reduced Latin squares

order	cyclic	non-cyclic group	non-group	all	equivalence classes
2	1	0	0	1	1
3	1	0	0	1	1
4	3	1	0	4	2
5	6	0	50	56	2

Numbers of reduced Latin squares

order	cyclic	non-cyclic group	non-group	all	equivalence classes
2	1	0	0	1	1
3	1	0	0	1	1
4	3	1	0	4	2
5	6	0	50	56	2
6	60	80	9268	9408	22

6: Frolov, 1890; Tarry, 1900; Fisher and Yates, 1934

Numbers of reduced Latin squares

order	cyclic	non-cyclic group	non-group	all	equivalence classes
2	1	0	0	1	1
3	1	0	0	1	1
4	3	1	0	4	2
5	6	0	50	56	2
6	60	80	9268	9408	22
7	120	0	16941960	16942080	564

6: Frolov, 1890; Tarry, 1900; Fisher and Yates, 1934
7: Frolov (wrong); Norton, 1939 (incomplete); Sade, 1948; Saxena, 1951

Numbers of reduced Latin squares

order	cyclic	non-cyclic group	non-group	all	equivalence classes
2	1	0	0	1	1
3	1	0	0	1	1
4	3	1	0	4	2
5	6	0	50	56	2
6	60	80	9268	9408	22
7	120	0	16941960	16942080	564
8	1260	1500	$> 10^{12}$	$> 10^{12}$	1676267

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9	6720	840	$> 10^{15}$	$> 10^{15}$	$> 10^{12}$
10	90720	36288	$> 10^{25}$	$> 10^{25}$	$> 10^{18}$
11	36288	0	$> 10^{34}$	$> 10^{34}$	$> 10^{26}$

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How to randomize? II

R. A. Fisher: *Statistical Methods for Research Workers*. Edinburgh, Oliver and Boyd, 1925.

F. Yates: The formation of Latin squares for use in field experiments. *Empire Journal of Experimental Agriculture*, **1** (1933), 235–244.

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These three all argued that randomization should ensure validity by eliminating bias in the estimation of the difference between the effect of any two treatments, and in the estimation of the variance of the foregoing estimator. This assumes that the data analysis allows for the effects of rows and columns.

Valid randomization

Random choice of a Latin square from a given set $\mathcal{L}$ of Latin squares or order n is valid if

- every cell in the square is equally likely to have each letter (this ensures lack of bias in the estimation of the difference between treatment effects)

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- every ordered pair of cells in different rows and columns has probability $1/n(n-1)$ of having the same specified letter,
  and probability $(n-2)/n(n-1)^2$ of having each ordered pair of distinct letters
  (this ensures lack of bias in the estimation of the variance).

Some methods of valid randomization

1. Permute rows by a random permutation and permute columns by an independently chosen random permutation (a.k.a. randomize rows and columns)—
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1. Permute rows by a random permutation and permute columns by an independently chosen random permutation (a.k.a. randomize rows and columns)—
   now the standard method.
2. Use any doubly transitive group in the above,
   rather than the whole symmetric group S_n
   (Grundy and Healy, 1950; Bailey, 1983).
3. Choose a Latin square at random
   from a complete set of mutually orthogonal Latin squares,
   and then randomize letters
   (Preece, Bailey and Patterson, 1978, following a 1935 remark of Fisher's when discussing a paper of Neyman).

Gerechte designs

Behrens introduced 'gerechte' designs in 1956.

A	B	C	E	D	F
D	E	F	B	C	A
B	C	E	F	A	D
F	D	A	C	B	E
C	F	D	A	E	B
E	A	B	D	F	C

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Randomize the 3 pairs of rows;
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F	*D*	*A*	*C*	*B*	*E*
C	*F*	*D*	*A*	*E*	*B*
E	*A*	*B*	*D*	*F*	*C*

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Randomize the 3 pairs of rows;
randomize the 2 rows within each pair; randomize the 2 triples of columns; randomize the 3 columns within each triple.
But then validity requires data analysis to allow for small rows and small columns, so the patterns in the small rows and small columns are a relevant part of the design.

Incomplete blocks

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How should we build incomplete-block designs?

Lattice designs for n^2 treatments in blocks of size n

F. Yates: A new method of arranging variety trials involving a large number of varieties. *Journal of Agricultural Science*, **26** (1936), 424–455.

Treatments

1	2	3
4	5	6
7	8	9

Latin square

A	B	C
C	A	B
B	C	A

Greek square

α	β	γ
β	γ	α
γ	α	β

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A design with 6 blocks of size 3 (shown as columns),

1	4	7	1	2	3
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β	γ	α
γ	α	β

A design with 6 blocks of size 3 (shown as columns),
or 9 blocks of size 3,

1	4	7	1	2	3	1	2	3
2	5	8	4	5	6	5	6	4
3	6	9	7	8	9	9	7	8

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Greek square

α	β	γ
β	γ	α
γ	α	β

A design with 6 blocks of size 3 (shown as columns), or 9 blocks of size 3, or 12 blocks of size 3.

1	4	7	1	2	3	1	2	3	1	2	3
2	5	8	4	5	6	5	6	4	6	4	5
3	6	9	7	8	9	9	7	8	8	9	7

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1	4	7	1	2	3	1	2	3	1	2	3
2	5	8	4	5	6	5	6	4	6	4	5
3	6	9	7	8	9	9	7	8	8	9	7

The last design is balanced because every pair of treatments occur together in the same number of blocks.

Now add four more treatments

1	4	7	1	2	3	1	2	3	1	2	3
2	5	8	4	5	6	5	6	4	6	4	5
3	6	9	7	8	9	9	7	8	8	9	7

Now add four more treatments

1	4	7	1	2	3	1	2	3	1	2	3
2	5	8	4	5	6	5	6	4	6	4	5
3	6	9	7	8	9	9	7	8	8	9	7
10	10	10									

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3	6	9	7	8	9	9	7	8	8	9	7
10	10	10	11	11	11						

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This design is also balanced.

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Balanced designs are optimal in the sense of minimizing variance (Kshirsagar, 1958).

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So are all these lattice designs (Cheng and Bailey, 1991).

Optimality was not really defined until the 1950s.

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The balanced designs are an affine plane and a projective plane. Yates did not know anything about such geometries in 1936.

A hypothetical cheese-tasting experiment

	Taster					
Order	1	2	3	4	5	6
1	*E*	*B*	*F*	*A*	*C*	*D*
2	*B*	*C*	*D*	*E*	*F*	*A*
3	*A*	*E*	*C*	*B*	*D*	*F*
4	*F*	*D*	*E*	*C*	*A*	*B*
5	*D*	*A*	*B*	*F*	*E*	*C*
6	*C*	*F*	*A*	*D*	*B*	*E*

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What happens if cheese E leaves a nasty after-taste?

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What happens if cheese *E* leaves a nasty after-taste?

Is this fair to cheese *B*?

Column-complete Latin squares

Definition

A Latin square is column-complete if each treatment is immediately followed, in the same column, by each other treatment exactly once.

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E. J. Williams: Experimental designs balanced for the estimation of residual effects of treatments. *Australian Journal of Scientific Research, Series A, Physical Sciences*, **2** (1949), 149–168.

0	1	2	3	4	5
1	2	3	4	5	0
5	0	1	2	3	4
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2	3	4	5	0	1
4	5	0	1	2	3
3	4	5	0	1	2

Williams gave a method of construction for all even orders. His squares are still widely used in tasting experiments and in trials of new drugs to alleviate symptoms of chronic conditions.

Complete Latin squares

A Latin square is complete if it is both row-complete and column-complete.

Quasi-complete Latin squares

For some experiments on the ground,
an East neighbour is as bad as a West neighbour,
and a South neighbour is as bad as a North neighbour.

Definition

A Latin square is quasi-complete if each treatment has
each other treatment next to it in the same row twice,
and next to it in the same column twice, in either direction.

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1	2	0	3	4
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2	3	1	4	0

Freeman (1979) defined these. Freeman (1981) gave the results of a computer enumeration for small orders. Bailey (1984) gave a method of construction for all orders.

A randomization paradox

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When $n = 7$, there is a set $\mathcal{L}_1$ of 864 such quasi-complete Latin squares that makes this randomization valid.

The set $\mathcal{L}_2$ of all known such quasi-complete Latin squares of order 7 contains 896 squares; random choice from this larger set is not valid.

Back to pairs of orthogonal Latin squares

Question (Euler, 1782)

For which values of n does there exist a pair of orthogonal Latin squares of order n?

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Proof.

(i) If n is odd, the Latin squares with entries in (i, j) defined by $i + j$ and $i + 2j$ modulo n are mutually orthogonal.

(ii) If $n = 4$ or $n = 8$ such a pair of squares can be constructed from a finite field.

(iii) If L_1 is orthogonal to L_2, where L_1 and L_2 have order n, and M_1 is orthogonal to M_2, where M_1 and M_2 have order m, then a product construction gives squares $L_1 \otimes M_1$ orthogonal to $L_2 \otimes M_2$, where these have order nm.

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Euler could not find a pair of orthogonal Latin squares of order 6, or 10, or ....

Euler's conjecture: order 6

On 10 August 1842, Heinrich Schumacher, the astronomer in Altona, wrote a letter to Gauß, telling him that his assistant, Thomas Clausen, had proved that there is no pair of orthogonal Latin squares of order 6.

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Theorem (Tarry, 1900)

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Theorem (Tarry, 1900)

There is no pair of orthogonal Latin squares of order 6.

Proof.

Exhaustive enumeration by hand, after dividing Latin squares of order 6 into 17 families. □

The end of the conjecture

Theorem (Bose and Shrikhande, 1959)

There is a pair of orthogonal Latin squares of order 22.

The end of the conjecture

Theorem (Bose and Shrikhande, 1959)

There is a pair of orthogonal Latin squares of order 22*.*

Theorem (Parker, 1959)

If $n = (3q - 1)/2$ *and*
q *is a power of an odd prime and* $q - 3$ *is divisible by* 4*,*
then there is a pair of orthogonal Latin squares of order n*.*
In particular, there are pairs of orthogonal Latin squares of orders 10*,* 34*,* 46 *and* 70*.*

The end of the conjecture

Theorem (Bose and Shrikhande, 1959)

There is a pair of orthogonal Latin squares of order 22*.*

Theorem (Parker, 1959)

If $n = (3q - 1)/2$ *and*
q *is a power of an odd prime and* $q - 3$ *is divisible by* 4*,*
then there is a pair of orthogonal Latin squares of order n*.*
In particular, there are pairs of orthogonal Latin squares of orders 10*,* 34*,* 46 *and* 70*.*

Theorem (Bose, Shrikhande and Parker, 1960)

If n *is not equal to* 2 *or* 6*,*
then there exists a pair of orthogonal Latin squares of order n*.*