Circular designs balanced for neighbours at distances one and two

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University of St Andrews, 24 April 2017

Joint work with Tank Aldred (University of Otago, New Zealand), Brendan McKay (ANU, Australia) and Ian Wanless (Monash University, Australia)

David Finney's 100th birthday cake, January 2017

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Take your knife and cut this into ten rows.

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(i) Each row has each of ten numbers (0–9) once.
Lay the rows out one after the other to give a sequence of 100 numbers.

(ii) Each ordered pair of numbers (0–9) occurs precisely once as ordered neighbours
(if we imagine that the last entry is repeated before the first entry).

Serially balanced sequences

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- found some for 2, 6, 7, 8, 9, 10, 11, 14, 18, 22
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Nonyane and Theobald (2007)

- described a computer algorithm which had succeeded in finding such a sequence for all values of n which had been tried, viz. $8, 9, \ldots, 34$.

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In a further variant, it is assumed that the left-neighbour effect is the same as the right-neighbour effect, so all that is needed is that each unordered pair of treatments occurs just once as neighbours and just once with a single unit in between.

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I shall report progress on finding methods of constructing the three types of design.

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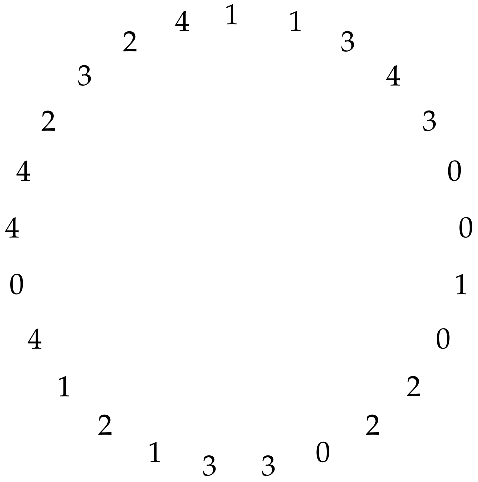
The marine biologist required that

(i) each ordered pair of items should occur just once as ordered neighbours around the circumference of the tank;

(ii) each ordered pair of items should occur just once with a single item in between them, in order.

A circular design for 5 treatments with neighbour balance at distances one and two



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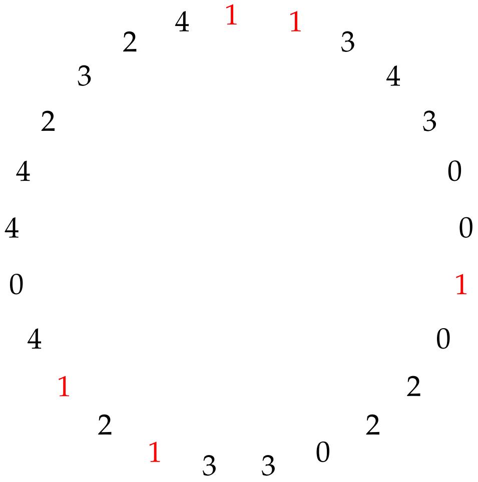

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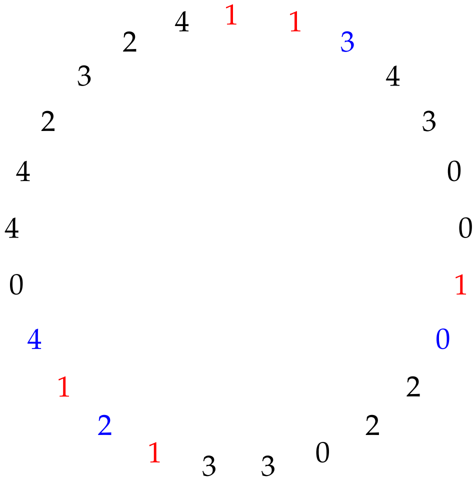

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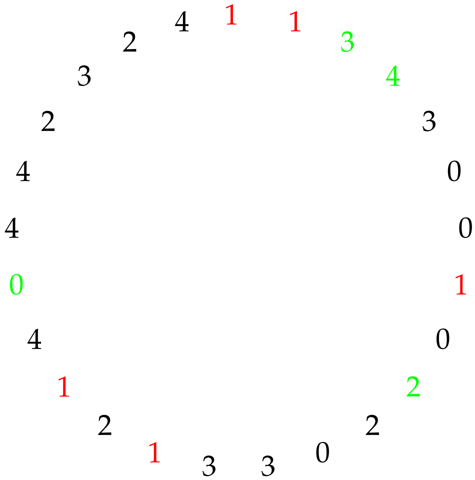
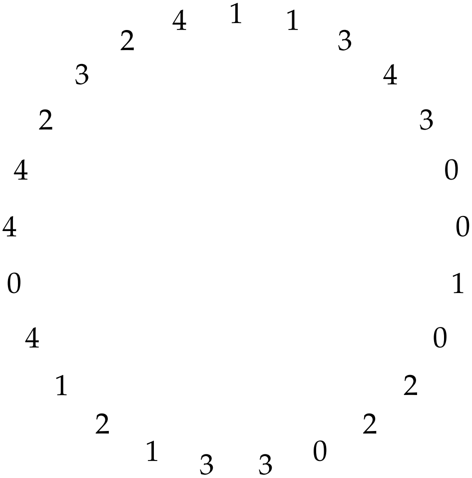
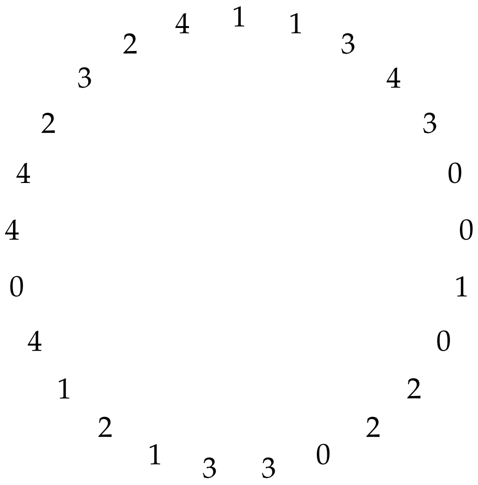




Look at the design while standing in the centre of the circle.
Then 'right neighbour' = 'clockwise neighbour'
and 'left neighbour' = 'anti-clockwise neighbour'.

The lazy way to write the design

(1 1 3 4 3 0 0 1 0 2 2 0 3 3 1 2 1 4 0 4 4 2 3 2 4)



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in other words, the design has neighbour balance at distances one and two.

Generalize the original problem

I wanted to prepare myself for future design requests like this.

Can we construct such a neighbour-balanced design
for n treatments each replicated n times
around a circle with space for n^2 items?

Those conditions again

Among the triples of the form

$$\big(\tau(i-1), \tau(i), \tau(i+1)\big),$$

each ordered pair of treatments occurs once in positions 1 and 2, once in positions 1 and 3, and once in positions 2 and 3.

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These are conditions for a Latin square
whose rows and columns have the same labels as the letters
—a quasigroup.

Building the design from a quasigroup (Latin square)

The quasigroup operation $\circ$ is defined by

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In the circular design, each triple should have the form

$$(a, b, a \circ b).$$

We can start with any ordered pair (x, y) and successively build the circular design from the quasigroup as

$$x \quad y \quad x \circ y \quad y \circ (x \circ y) \quad (x \circ y) \circ (y \circ (x \circ y)) \quad \cdots$$

Latin square to circle

$\circ$	A	B	C	D
A	B	A	D	C
B	C	D	A	B
C	D	C	B	A
D	A	B	C	D

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This quasigroup gives a design with four separate circles, not one.

$$(\; A \quad A \quad B \quad A \quad C \quad D \;)$$

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2	3	4	0	2	1
3	0	2	4	1	3
4	4	1	3	0	2

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BUT we have been unable to prove that they always exist.

It is quite easy to show that, if $Q = \mathbb{Z}_{p^s}$ or $Q = \mathrm{GF}(p^s)$,
then no binary operation of the form

$$x \circ y = ax + by + c$$

makes Q into an Eulerian quasigroup.

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In September 2004 I spent two weeks at ANU working with BDM and IMW (and remotely with RELA). We solved the two variants completely.

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Preece (1975 ACC, Adelaide) showed that, for overall balance, the missing pairs at distance two must also be the self-pairs.

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Our circular design is equivalent to an idempotent quasigroup in which the $n(n-1)$ off-diagonal cells give a single circle.

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The treatments are the integers modulo 5, together with ∞.

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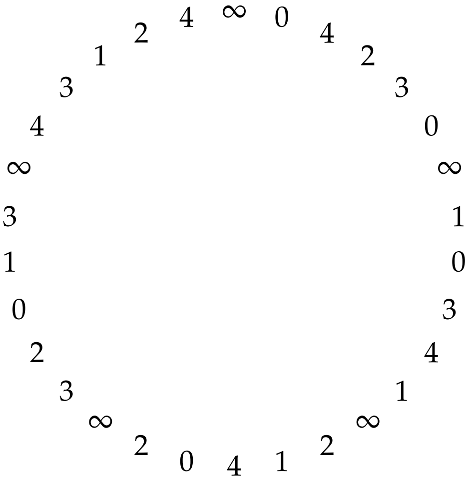

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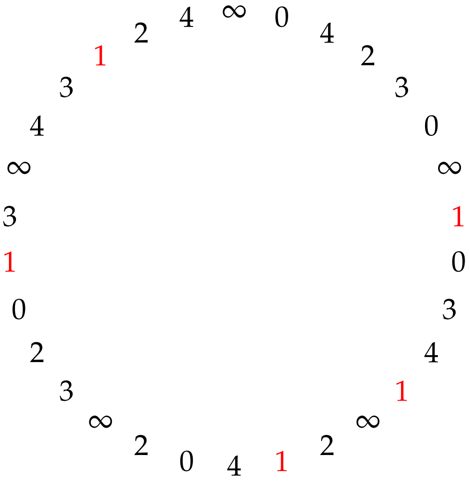

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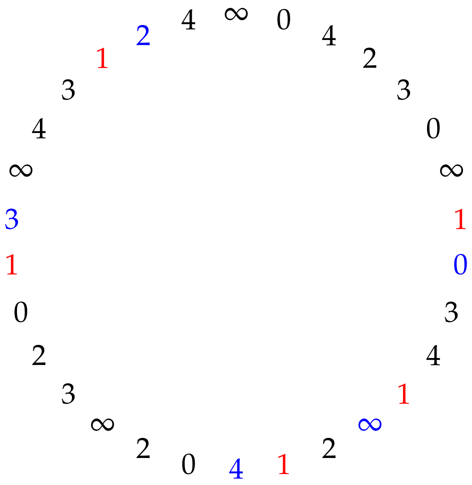

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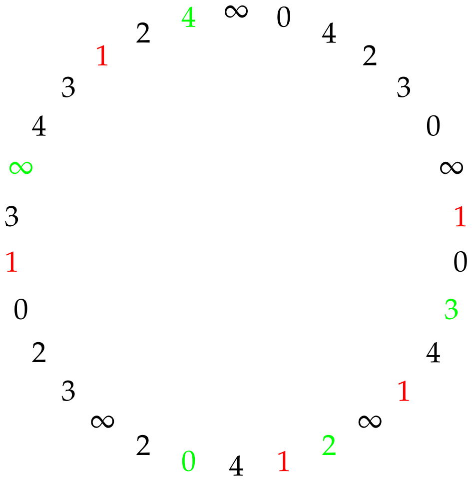

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$$\text{Small thing} \xrightarrow{\quad\quad\quad\text{algorithm}\quad\quad\quad} \text{Large thing}$$

Paradigm

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Paradigm

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Any triple (a, b, a) gives b as a neighbour of a on both sides, so there can be no such triples.

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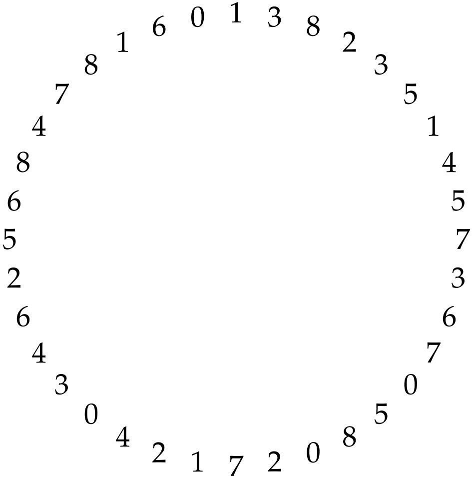
A circular design for 9 treatments with undirectional neighbour balance at distances one and two



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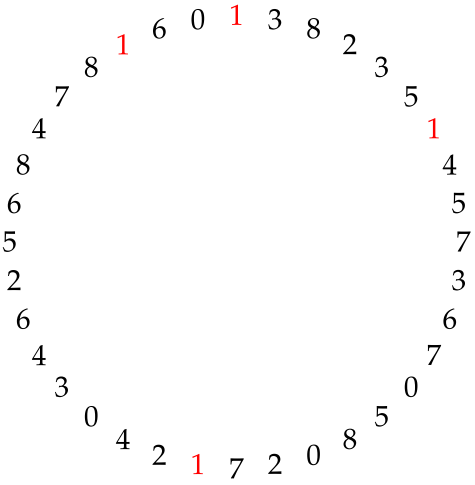

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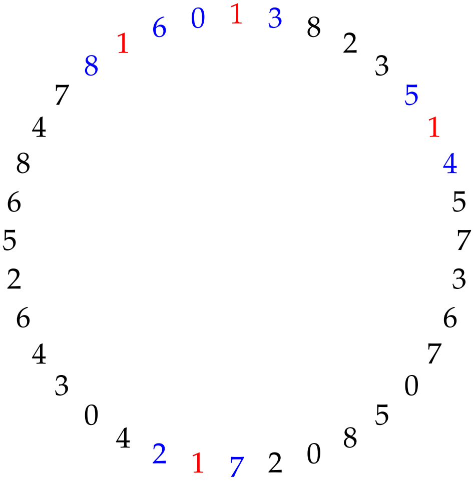

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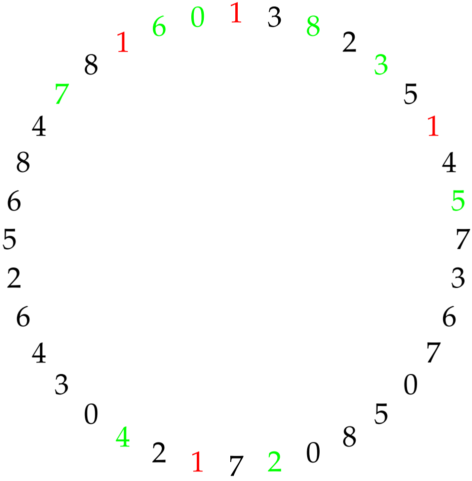

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Solution for variant II

Theorem

Given an initial circular sequence of $(n-1)/2$ of the integers modulo n satisfying those conditions, that construction always produces a circular sequence balanced for undirected neighbours at distances one and two.

Theorem

Such an initial sequence can be constructed whenever n is odd and $n \geq 9$. There is also such a circular sequence when $n = 7$.

Back to the original question

A quasigroup of order n with operation $\circ$ is Eulerian if the sequence

$$x \quad y \quad x \circ y \quad y \circ (x \circ y) \quad (x \circ y) \circ (y \circ (x \circ y)) \quad \cdots$$

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Conjecture

If $n \geq 5$ then there exists an Eulerian quasigroup of order n.

Coprime sizes

Theorem

If $(Q_1, \bullet)$ *and* $(Q_2, \circ)$ *are Eulerian quasigroups of orders* n *and* m*, where* n *and* m *are coprime, then* $Q_1 \otimes Q_2$ *is an Eulerian quasigroup of order* nm*.*

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Proof.

In the sequence

$$(a, x) \quad (b, y) \quad (a \bullet b, x \circ y) \quad (b \bullet (a \bullet b), y \circ (x \circ y)) \quad \cdots$$

the first coordinates repeat every n^2 steps, but not earlier, and the second coordinates repeat every m^2 steps, but not earlier. □

Some more history

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We still have no general construction,
but a paper eventually got written and submitted.

Strategy

Because of the ‘coprime’ theorem, and because there is no solution for 2, 3 or 4, all we have to do is to find an Eulerian quasigroup for all of the following orders:

- q where q is an odd prime power and $q \geq 5$

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(and the paper had been accepted before we realised that we also need)

- $3 \times$ all non-trivial powers of 2.

Reminder: the obvious way is no good

If p is prime and $Q = \mathbb{Z}_p$, then no binary operation of the form

$$x \circ y = ax + by + c$$

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If $a + b - 1 = 0$ and $b \neq 2$ and $t = -(b - 2)^{-1}c$
then $mt \circ (m + 1)t = (m + 2)t$ for all integers m,
so we get a circle of size p.

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If $a + b - 1 = 0$ and $b = 2$
then ${}^{m}\mathrm{C}_2 c \circ {}^{m+1}\mathrm{C}_2 c = {}^{m+2}\mathrm{C}_2 c$ for all positive integers m,
so we get a circle of size p.

Technique to avoid brute search

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For example, when $q = 7$ put $\pi = (0\,1\,2)(3\,4)$ so that

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This
(the permutation $(0\ 1\ 2)$ with some adjacent transpositions)
works for all odd numbers that we have tried.

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Theorem

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Theorem

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... so IMW found another technique to cut down the computer search when n is even.

. . . for all practical purposes

Theorem
If $n \geq 5$ and there is no Eulerian quasigroup of order n then n is divisible by a prime power exceeding 1000.

...for all practical purposes

Theorem
If $n \geq 5$ and there is no Eulerian quasigroup of order n then n is divisible by a prime power exceeding 1000.

But, just as for the problem with serially balanced sequences,
we do not have a general construction and
we do not have a proof that they exist for all large enough n.