Latin squares: Some history, with an emphasis on their use in designed experiments

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Abstract

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Fisher and Neyman had a famous falling out over Latin squares in 1935 when Neyman proved that use of Latin squares in experiments gives biased results. A six-week international workshop in Boulder, Colorado in 1957 resolved this, but the misunderstanding surfaced again in a Statistics paper published in 2017.

What is a Latin square?

Definition

Let n be a positive integer.
A Latin square of order n is an $n \times n$ array of cells in which n symbols are placed, one per cell, in such a way that each symbol occurs once in each row and once in each column.

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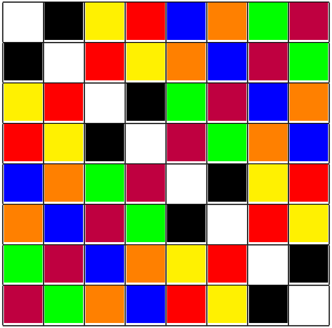
The symbols may be letters, numbers, colours, ...

A Latin square of order 8



A Latin square of order 6

E	B	F	A	C	D
B	C	D	E	F	A
A	E	C	B	D	F
F	D	E	C	A	B
D	A	B	F	E	C
C	F	A	D	B	E

A stained glass window in Caius College, Cambridge

photograph by
J. P. Morgan

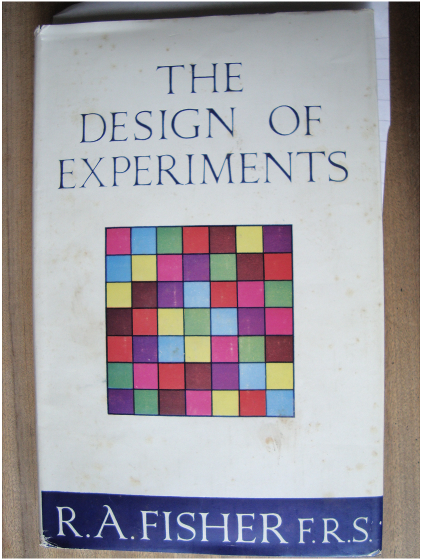
And on the opposite side of the hall

And on the opposite side of the hall

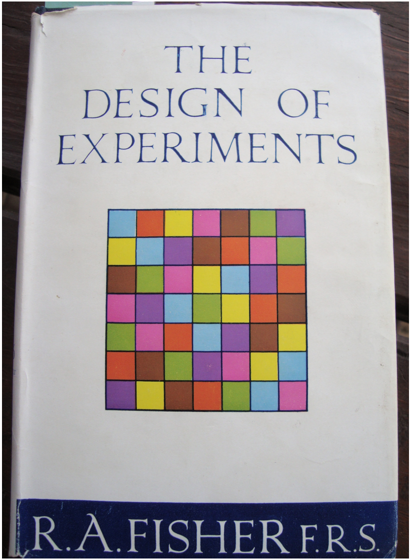
R. A. Fisher promoted the use of Latin squares in experiments while at Rothamsted (1919–1933) and his 1935 book *The Design of Experiments*.

Stained glass window; book cover; INI logo

Latin squares on book covers

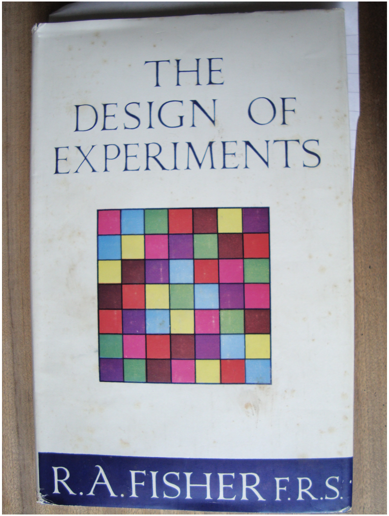


6th edition



7th edition

A Latin square of order 7



This Latin square was on the cover of the first edition of *The Design of Experiments*.

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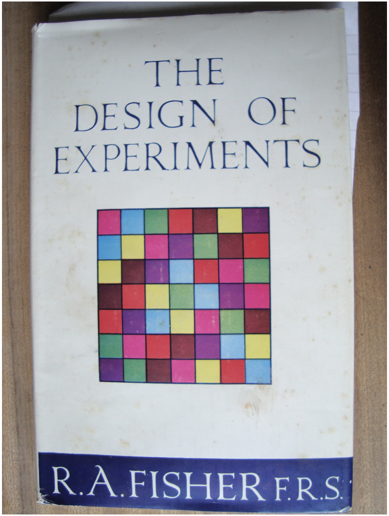

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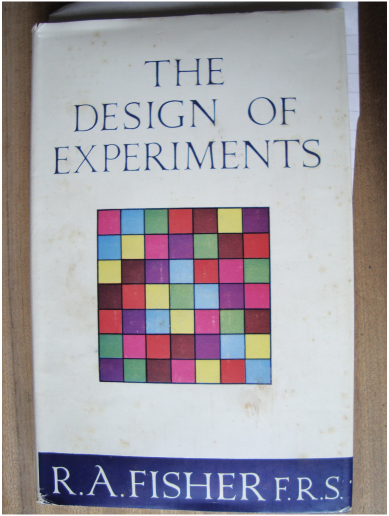

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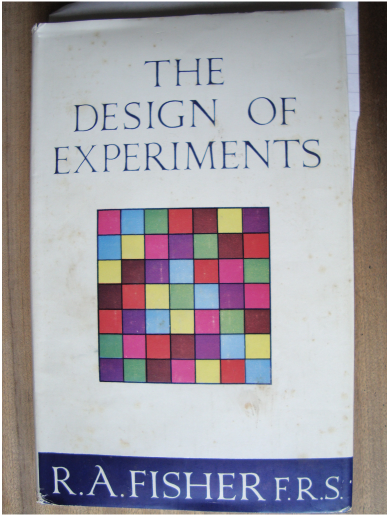

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Why is it called 'Latin'?

What are Latin squares used for?

Agricultural field trials, with rows and columns corresponding to actual rows and columns on the ground (possibly the width of rows is different from the width of columns).

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"…on any given field agricultural operations, at least for centuries, have followed one of two directions, which are usually those of the rows and columns; consequently streaks of fertility, weed infestation, etc., do, in fact, occur predominantly in those two directions."

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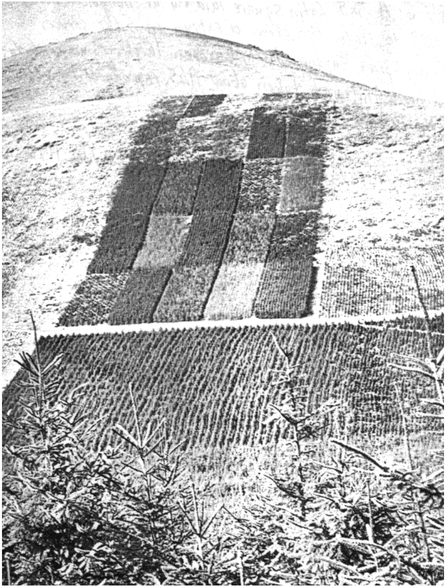
This assumption is dubious for field trials in Australia.

An experiment on potatoes at Ely in 1932

E	B	F	A	C	D
B	C	D	E	F	A
A	E	C	B	D	F
F	D	E	C	A	B
D	A	B	F	E	C
C	F	A	D	B	E

Treatment	A	B	C	D	E	F
Extra nitrogen	0	0	0	1	1	1
Extra phosphate	0	1	2	0	1	2

A forestry experiment



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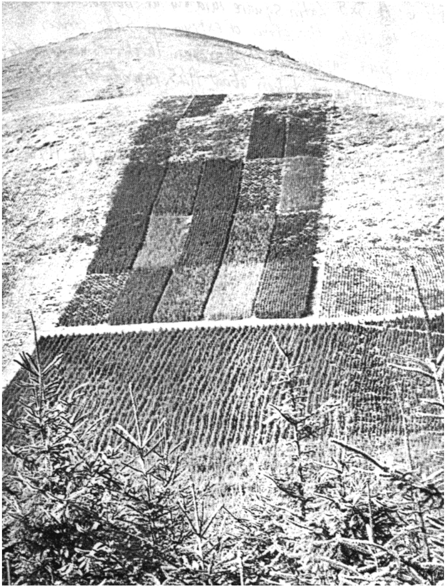

Experiment on a hillside near Beddgelert Forest, designed by Fisher and laid out in 1929

Other sorts of rows and columns: animals

An experiment on 16 sheep carried out by François Cretté de Palluel, reported in *Annals of Agriculture* in 1790. They were fattened on the given diet, and slaughtered on the date shown.

slaughter date	Breed: Ile de France	Beauce	Champagne	Picardy
20 Feb	potatoes	turnips	beets	oats & peas
20 Mar	turnips	beets	oats & peas	potatoes
20 Apr	beets	oats & peas	potatoes	turnips
20 May	oats & peas	potatoes	turnips	beets

Other sorts of rows and columns: plants in pots

An experiment where treatments can be applied to individual leaves of plants in pots.

	plant			
height	1	2	3	4
1	A	B	C	D
2	B	A	D	C
3	C	D	A	B
4	D	C	B	A

Graeco-Latin squares

A	B	C
C	A	B
B	C	A

α	β	γ
β	γ	α
γ	α	β

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When the two Latin squares are superposed,
each Latin letter occurs exactly once with each Greek letter.

A α	B β	C γ
C β	A γ	B α
B γ	C α	A β

Graeco-Latin squares

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$A\ \alpha$	$B\ \beta$	$C\ \gamma$
$C\ \beta$	$A\ \gamma$	$B\ \alpha$
$B\ \gamma$	$C\ \alpha$	$A\ \beta$

Euler called such a superposition a 'Graeco-Latin square'.

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$B\ \gamma$	$C\ \alpha$	$A\ \beta$

Euler called such a superposition a 'Graeco-Latin square'.
The name 'Latin square' seems to be a back-formation from this.

Pairs of orthogonal Latin squares

Definition
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We have just seen a pair of orthogonal Latin squares of order 3.

Mutually orthogonal Latin squares

Definition

A collection of Latin squares of the same order is **mutually orthogonal** if every pair is orthogonal.

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A collection of Latin squares of the same order is mutually orthogonal if every pair is orthogonal.

Example ($n = 4$)

$A\alpha 1$	$B\beta 2$	$C\gamma 3$	$D\delta 4$
$B\gamma 4$	$A\delta 3$	$D\alpha 2$	$C\beta 1$
$C\delta 2$	$D\gamma 1$	$A\beta 4$	$B\alpha 3$
$D\beta 3$	$C\alpha 4$	$B\delta 1$	$A\gamma 2$

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$C\delta 2$	$D\gamma 1$	$A\beta 4$	$B\alpha 3$
$D\beta 3$	$C\alpha 4$	$B\delta 1$	$A\gamma 2$

Theorem

If there exist k mutually orthogonal Latin squares $L_1, \ldots, L_k$ of order n, then $k \leq n - 1$.

When is the maximum achieved?

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For example, $n = 2, 3, 4, 5, 7, 8, 9, 11, 13, \ldots$.

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R. A. Fisher and F. Yates: *Statistical Tables for Biological, Agricultural and Medical Research*. Edinburgh, Oliver and Boyd, 1938.
This book gives a set of $n-1$ MOLS for $n = 3, 4, 5, 7, 8$ and 9. The set of order 9 is not made by the usual finite-field construction, and it is not known how Fisher and Yates obtained this.

An industrial experiment using MOLS

L. C. H. Tippett: Applications of statistical methods to the control of quality in industrial production. Manchester Statistical Society (1934). (Cited by Fisher, 1935)

A cotton mill has 5 spindles, each made of 4 components. Why is one spindle producing defective weft?

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Period	i	ii	iiii	iv	v
1	$A\alpha 1$	$B\beta 2$	$C\gamma 3$	$D\delta 4$	$E\varepsilon 5$
2	$E\delta 3$	$A\varepsilon 4$	$B\alpha 5$	$C\beta 1$	$D\gamma 2$
3	$D\beta 5$	$E\gamma 1$	$A\delta 2$	$B\varepsilon 3$	$C\alpha 4$
4	$C\varepsilon 2$	$D\alpha 3$	$E\beta 4$	$A\gamma 5$	$B\delta 1$
5	$B\gamma 4$	$C\delta 5$	$D\varepsilon 1$	$E\alpha 2$	$A\beta 3$

1st component	2nd component	3rd component	4th component
i–v	A–E	α–ε	1–5

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3	$D\beta 5$	$E\gamma 1$	$A\delta 2$	$B\varepsilon 3$	$C\alpha 4$
4	$C\varepsilon 2$	$D\alpha 3$	$E\beta 4$	$A\gamma 5$	$B\delta 1$
5	$B\gamma 4$	$C\delta 5$	$D\varepsilon 1$	$E\alpha 2$	$A\beta 3$

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How to randomize? I

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> Systematic arrangements in a square ... have been used previously for variety trials in, for example, Ireland and Denmark; but the term "Latin square" should not be applied to any such systematic arrangements. The problem of the Latin Square, from which the name was borrowed, as formulated by Euler, consists in the enumeration of *every possible* arrangement, subject to the conditions that each row and each column shall contain one plot of each variety. Consequently, the term Latin Square should only be applied to a process of randomization by which one is selected at random out of the total number of Latin Squares possible, ...

How many different Latin squares of order n are there?

Are these two Latin squares the same?

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C	A	B
B	C	A

1	2	3
3	1	2
2	3	1

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3	1	2
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To answer this question, we will have to insist that all the Latin squares use the same symbols, such as $1, 2, \ldots, n$.

Reduced Latin squares, and equivalence

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A Latin square is reduced if the symbols in the first row and first column are $1, 2, \ldots, n$ in natural order.

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Definition

Latin squares L and M are equivalent if there is
a permutation f of the rows, a permutation g of the columns
and permutation h of the symbols such that

$$\begin{array}{cccccccc} \text{symbol} & s & \text{is in row} & r & \text{and column} & c & \text{of} & L \\ & & & \Longleftrightarrow & & & & \\ \text{symbol} & h(s) & \text{is in row} & f(r) & \text{and column} & g(c) & \text{of} & M. \end{array}$$

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Theorem

If there are m reduced squares in an equivalence class of Latin squares of order n, then the total number of Latin squares in the equivalence class is $m \times n! \times (n-1)!$.

Numbers of reduced Latin squares

order	cyclic	non-cyclic group	non-group	all	equivalence classes
2	1	0	0	1	1
3	1	0	0	1	1
4	3	1	0	4	2
5	6	0	50	56	2

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3	1	0	0	1	1
4	3	1	0	4	2
5	6	0	50	56	2
6	60	80	9268	9408	22

6: Frolov, 1890; Tarry, 1900; Fisher and Yates, 1934

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6	60	80	9268	9408	22
7	120	0	16941960	16942080	564

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7: Frolov (badly wrong); Norton, 1939 (omitted one species); Sade, 1948; Saxena, 1951

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9	6720	840	$> 10^{15}$	$> 10^{15}$	$> 10^{12}$

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10	90720	36288	$> 10^{25}$	$> 10^{25}$	$> 10^{18}$

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11	36288	0	$> 10^{34}$	$> 10^{34}$	$> 10^{26}$

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How to randomize? II

R. A. Fisher: *Statistical Methods for Research Workers*. Edinburgh, Oliver and Boyd, 1925.

F. Yates: The formation of Latin squares for use in field experiments. *Empire Journal of Experimental Agriculture*, **1** (1933), 235–244.

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These three all argued that randomization should ensure validity by eliminating bias in the estimation of the difference between the effect of any two treatments, and in the estimation of the variance of the foregoing estimator. This assumes that the data analysis allows for the effects of rows and columns.

Valid randomization

Random choice of a Latin square from a given set $\mathcal{L}$ of Latin squares or order n is valid if

- every cell in the square is equally likely to have each letter (this ensures lack of bias in the estimation of the difference between treatment effects)

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- every cell in the square is equally likely to have each letter (this ensures lack of bias in the estimation of the difference between treatment effects)
- every ordered pair of cells in different rows and columns has probability $1/n(n-1)$ of having the same specified letter,
  and probability $(n-2)/n(n-1)^2$ of having each ordered pair of distinct letters
  (this ensures lack of bias in the estimation of the variance).

Some methods of valid randomization

1. Permute rows by a random permutation and permute columns by an independently chosen random permutation (a.k.a. randomize rows and columns)—
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(Grundy and Healy, 1950; Bailey, 1983).

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now the standard method.
2. Use any doubly transitive group in the above,
rather than the whole symmetric group S_n
(Grundy and Healy, 1950; Bailey, 1983).
3. Choose a Latin square at random
from a complete set of mutually orthogonal Latin squares,
and then randomize letters
(Preece, Bailey and Patterson, 1978, following a 1935 remark of Fisher's when discussing a paper of Neyman).

Back to pairs of orthogonal Latin squares

Question (Euler, 1782)

For which values of n does there exist a pair of orthogonal Latin squares of order n?

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If n is odd, or if n is divisible by 4,
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If n is odd, or if n is divisible by 4, then there is a pair of orthogonal Latin squares of order n.

Proof.

(i) If n is odd, the Latin squares with entries in (i, j) defined by $i + j$ and $i + 2j$ modulo n are mutually orthogonal.

(ii) If $n = 4$ or $n = 8$ such a pair of squares can be constructed from a finite field.

(iii) If L_1 is orthogonal to L_2, where L_1 and L_2 have order n, and M_1 is orthogonal to M_2, where M_1 and M_2 have order m, then a product construction gives squares $L_1 \otimes M_1$ orthogonal to $L_2 \otimes M_2$, where these have order nm.

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Euler could not find a pair of orthogonal Latin squares of order 6, or 10, or ....

Euler's conjecture: order 6

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Proof.

Exhaustive enumeration by hand, after dividing Latin squares of order 6 into 17 families. □

The end of the conjecture

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If $n = (3q - 1)/2$ *and*
q *is a power of an odd prime and* $q - 3$ *is divisible by* 4,
then there is a pair of orthogonal Latin squares of order n.
In particular, there are pairs of orthogonal Latin squares of orders 10, 34, 46 *and* 70.

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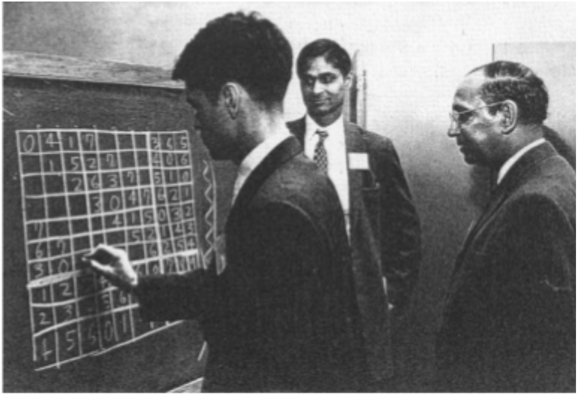
Theorem (Bose, Shrikhande and Parker, 1960)

If n *is not equal to* 2 *or* 6*,*
then there exists a pair of orthogonal Latin squares of order n*.*

New York Times, 16 April 1959

Major Mathematical Conjecture Propounded 177 Years Ago Is Disproved



(Copied from *The history of latin squares* by Lars Døvling Andersen, 2007)

Some problems with Fisher's exposition

Fisher was rather authoritarian about his work.
(Ironically, he may have inadvertently mimicked Karl Pearson.)
He liked to lay down the law before the law was properly formulated and understood. But

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- he rarely wrote down explicit formulae for his assumptions or methods
  (Frank Yates, his junior colleague then long-term successor at Rothamsted, did this very clearly, apparently with Fisher's blessing);
- some of his eye-catching early examples were inconsistent with his later developments
  (the lady tasting tea, and comments on an experiment of Darwin's (both in *Design of Experiments*, 1935)
  led to the randomization test,
  which he explicitly recanted in the 7th edition in 1960).

Explicit assumptions

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- τ_i depends only on treatment i,
  and we want to estimate differences like $\tau_1 - \tau_2$;
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Conclusions cannot be extrapolated.

The Fisher–Neyman row

Neyman read a paper on *Statistical problems in agricultural experimentation* to the Royal Statistical Society in 1935.
In this, he claimed to have proved that any experiment designed as a Latin square gives biased results
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Neyman moved to the USA, where Wilk and Kempthorne (ex-Rothamsted) developed his argument further in 1957.

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In later years, Kempthorne (who could be as rude as Fisher in writing but as nice as pie in person) also used the additive model. In a 1975 paper he went so far as to say that Neyman's null hypothesis (that $\sum_\omega Y_\omega(i)$ is the same for every treatment i) "is not scientifically relevant".

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Deng's response concluded

> *... as an assistant professor in the department founded by Neyman, I feel obligated to use it to continue the Neyman tradition.*