Balanced colourings and equitable partitions of triangular association schemes

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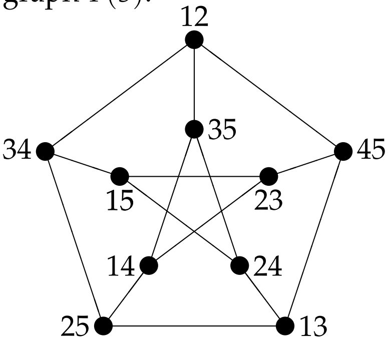

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$\circ$ = vertices joined to vertex $\{3, 5\}$

Commutative linear algebra

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The real vector space $\mathbb{R}^\Omega$ is the orthogonal direct sum of subspaces W_0, W_1 and W_2, each of which is (contained in) an eigenspace of A and an eigenspace of J, where W_0 is the one-dimensional subspace spanned by the all-1 vector $\mathbf{u}$.

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(I will identify W_1 and W_2 later.)

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The second is called an equitable partition of the graph.

Design question and statistical issues

We have a set $\mathcal{T}$ of t treatments. We need to choose a design, which is a function $f\colon \Omega \to \mathcal{T}$ allocating treatment $f(\omega)$ to experimental unit ω. How should we choose f?

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We do not know the values of γ_0, γ_1 and γ_2 in advance.
When is the choice of best design not affected by the values of γ_0, γ_1 and γ_2?

First desirable statistical condition

Condition 1 We want the variance V_{ij} of the estimator of $\tau_i - \tau_j$ to be the same for all pairs $\{i, j\}$ of distinct treatments.

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Apologies for the confusing notation.
For this combinatorial structure, i and j denote individuals, so treatments are usually denoted A, B, $\ldots$.

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If m is odd and $t = m$ we can do this by using a symmetric, idempotent Latin square of order m and omitting the main diagonal and plots above the main diagonal (idempotent means that this diagonal contains each letter once).
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Then each treatment occurs on $(m-1)/2$ plots, and $\lambda = m - 2$.
In fact, each treatment misses one individual and occurs once with every other individual.

An example with $m = 7$

	1	2	3	4	5	6
2	B					
3	C	D				
4	D	E	F			
5	E	F	G	A		
6	F	G	A	B	C	
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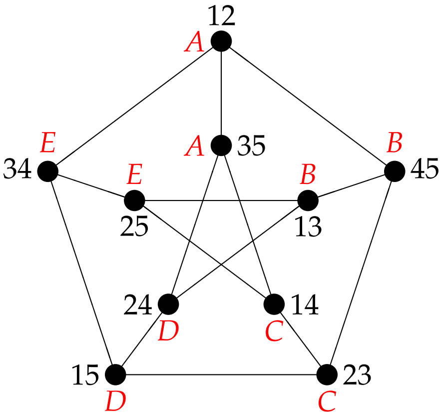
For strongly regular graphs in general, such designs are called balanced colourings of strongly regular graphs.

This design on the Petersen graph



For each treatment, there is one edge that has that treatment on both vertices.
For each pair of distinct treatments, there is one edge that has them on its endpoints.

Second desirable statistical condition

For $i = 1, \ldots, m$, let $\mathbf{v}_i$ be the vector taking the value 1 on each pair that includes individual i and value 0 elsewhere. Let V_{ind} be the m-dimensional subspace of $\mathbb{R}^\Omega$ spanned by $\mathbf{v}_1, \ldots, \mathbf{v}_m$.

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Condition 2 We want the linear combination of the Y_ω (for $\omega \in \Omega$) which gives the best estimate of $\tau_i - \tau_j$ (correct on average, smallest variance) to be the same as the best estimator when $\gamma_0 = \gamma_1 = \gamma_2$. This is the difference between the averages for plots with treatment i and those with treatment j.

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Solution The subspace V_T of $\mathbb{R}^\Omega$ consisting of vectors which are constant on each treatment can be orthogonally decomposed as

$$W_0 \oplus (V_T \cap W_1) \oplus (V_T \cap W_2).$$

Second desirable statistical condition, continued

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(a) $V_T \leq W_0 \oplus W_2$.

(b) $V_T \leq W_0 \oplus W_1$.

(c) $V_T \cap W_1$ and $V_T \cap W_2$ are both non-zero, and $V_T = W_0 \oplus (V_T \cap W_1) \oplus (V_T \cap W_2)$.

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(Start with a Latin square of the previous type; add an extra row at the bottom; move every diagonal element down to the bottom row; then put a dummy like ∞ on every diagonal cell.)

An example with $m = 8$

	1	2	3	4	5	6	7
2	C						
3	D	E					
4	E	F	G				
5	F	G	A	B			
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Each treatment occurs exactly once with each individual.

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Any subset of treatments may be merged into a single treatment.

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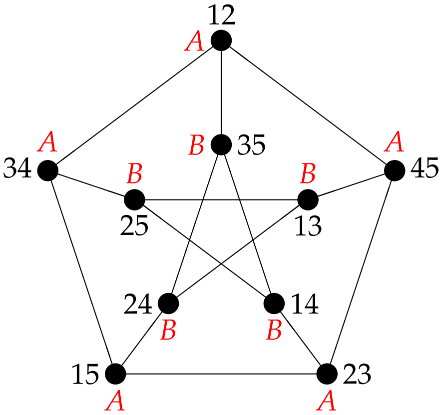
When $m = 9$ this gives

	1	2	3	4	5	6	7	8
2	1							
3	2	1						
4	3	2	1					
5	4	3	2	1				
6	4	4	3	2	1			
7	3	4	4	3	2	1		
8	2	3	4	4	3	2	1	
9	1	2	3	4	4	3	2	1

Solution (a) for Condition 2 when $m = 5$



Here A represents ± 1 mod 5 and B represents ± 2 mod 5.

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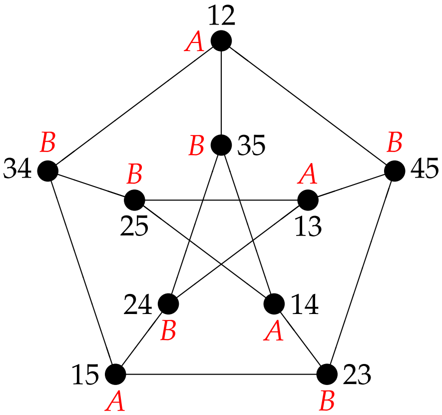

The two treatments are not equally replicated.

Solution (c) for Condition 2

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- If $i < j$ then let t_{ij} be any common divisor of s_i and s_j. Make a set $\mathcal{T}_{ij}$ of t_{ij} treatments. Allocate these to the cells in the rectangle $\mathcal{S}_j \times \mathcal{S}_i$ in such a way that all treatments appear equally often in each row and equally often in each column.

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Theorem about this solution

Theorem

For $i = 1, \ldots, n$,
let $\mathbf{w}_i$ be the vector whose entries are

$$\begin{cases} 0 & \textit{on all pairs which do not involve an individual of sort } i \\ 1 & \textit{on all pairs which involve a single individual of sort } i \\ 2 & \textit{on all pairs which involve two indiviudals of sort } i \end{cases}$$

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Then

- *The vectors $\mathbf{w}_1, \ldots, \mathbf{w}_n$ span an n-dimensional subspace of $V_T \cap (W_0 \oplus W_1)$.*
- *If $\mathbf{v} \in V_T$ is orthogonal to $\mathbf{w}_i$ for $i = 1, \ldots, n$ then $\mathbf{v} \in W_2$.*

An example with two sorts

Here $m = 9, n = 2, s_1 = 3, s_2 = 6$ and $t = 9$.

	1	2	3	4	5	6	7	8
2	A							
3	A	A						
4	B	C	D					
5	B	C	D	E				
6	D	B	C	F	I			
7	D	B	C	G	H	E		
8	C	D	B	H	F	G	I	
9	C	D	B	I	G	H	F	E

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	1	2	3	4	5	6	7	8
2	A							
3	A	A						
4	B	C	D					
5	B	C	D	E				
6	D	B	C	F	I			
7	D	B	C	G	H	E		
8	C	D	B	H	F	G	I	
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$\mathcal{S}_1 = \{1,2,3\}$, $\mathcal{T}_1 = \{A\}$ and $t_1 = 1$.

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	1	2	3	4	5	6	7	8
2	A							
3	A	A						
4	B	C	D					
5	B	C	D	E				
6	D	B	C	F	I			
7	D	B	C	G	H	E		
8	C	D	B	H	F	G	I	
9	C	D	B	I	G	H	F	E

$\mathcal{S}_1 = \{1, 2, 3\}$, $\mathcal{T}_1 = \{A\}$ and $t_1 = 1$.
$\mathcal{S}_2 = \{4, 5, 6, 7, 8, 9\}$, $\mathcal{T}_2 = \{E, F, G, H, I\}$ and $t_2 = 5$.

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Here $m = 9, n = 2, s_1 = 3, s_2 = 6$ and $t = 9$.

	1	2	3	4	5	6	7	8
2	A							
3	A	A						
4	B	C	D					
5	B	C	D	E				
6	D	B	C	F	I			
7	D	B	C	G	H	E		
8	C	D	B	H	F	G	I	
9	C	D	B	I	G	H	F	E

$\mathcal{S}_1 = \{1,2,3\}$, $\mathcal{T}_1 = \{A\}$ and $t_1 = 1$.
$\mathcal{S}_2 = \{4,5,6,7,8,9\}$, $\mathcal{T}_2 = \{E,F,G,H,I\}$ and $t_2 = 5$.
$\mathcal{T}_{12} = \{B,C,D\}$ and $t_{12} = 3$.

An example with three sorts

Here $m = 9$, $n = 3$, $s_1 = 1$, $s_2 = 4$, $s_3 = 4$ and $t = 12$.

	1	2	3	4	5	6	7	8
2	A							
3	A	B						
4	A	C	D					
5	A	D	C	B				
6	E	F	G	H	I			
7	E	G	H	I	F	J		
8	E	H	I	F	G	K	L	
9	E	I	F	G	H	L	K	J

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Here $m = 9$, $n = 3$, $s_1 = 1$, $s_2 = 4$, $s_3 = 4$ and $t = 12$.

	1	2	3	4	5	6	7	8
2	A							
3	A	B						
4	A	C	D					
5	A	D	C	B				
6	E	F	G	H	I			
7	E	G	H	I	F	J		
8	E	H	I	F	G	K	L	
9	E	I	F	G	H	L	K	J

$\mathcal{S}_1 = \{1\}$, $\mathcal{T}_1 = \emptyset$ and $t_1 = 0$.

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Here $m = 9$, $n = 3$, $s_1 = 1$, $s_2 = 4$, $s_3 = 4$ and $t = 12$.

	1	2	3	4	5	6	7	8
2	A							
3	A	B						
4	A	C	D					
5	A	D	C	B				
6	E	F	G	H	I			
7	E	G	H	I	F	J		
8	E	H	I	F	G	K	L	
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$\mathcal{S}_1 = \{1\}$, $\mathcal{T}_1 = \emptyset$ and $t_1 = 0$.
$\mathcal{S}_2 = \{2, 3, 4, 5\}$, $\mathcal{T}_2 = \{B, C, D\}$ and $t_2 = 3$.

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Here $m = 9$, $n = 3$, $s_1 = 1$, $s_2 = 4$, $s_3 = 4$ and $t = 12$.

	1	2	3	4	5	6	7	8
2	A							
3	A	B						
4	A	C	D					
5	A	D	C	B				
6	E	F	G	H	I			
7	E	G	H	I	F	J		
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$\mathcal{S}_3 = \{6,7,8,9\}$, $\mathcal{T}_3 = \{J,K,L\}$ and $t_3 = 3$.

An example with three sorts

Here $m = 9$, $n = 3$, $s_1 = 1$, $s_2 = 4$, $s_3 = 4$ and $t = 12$.

	1	2	3	4	5	6	7	8
2	*A*							
3	*A*	*B*						
4	*A*	*C*	*D*					
5	*A*	*D*	*C*	*B*				
6	*E*	*F*	*G*	*H*	*I*			
7	*E*	*G*	*H*	*I*	*F*	*J*		
8	*E*	*H*	*I*	*F*	*G*	*K*	*L*	
9	*E*	*I*	*F*	*G*	*H*	*L*	*K*	*J*

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Here $m = 9$, $n = 3$, $s_1 = 1$, $s_2 = 4$, $s_3 = 4$ and $t = 12$.

	1	2	3	4	5	6	7	8
2	A							
3	A	B						
4	A	C	D					
5	A	D	C	B				
6	E	F	G	H	I			
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	1	2	3	4	5	6	7	8
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3	A	B						
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5	A	D	C	B				
6	E	F	G	H	I			
7	E	G	H	I	F	J		
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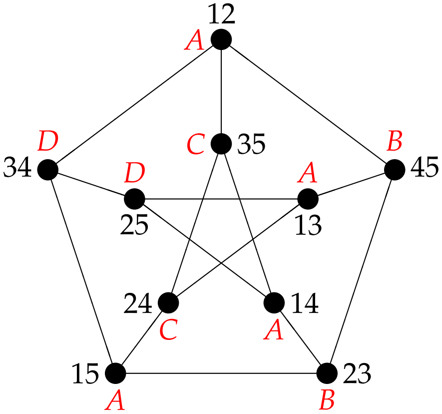
$\mathcal{S}_3 = \{6,7,8,9\}$, $\mathcal{T}_3 = \{J,K,L\}$ and $t_3 = 3$.
$\mathcal{T}_{12} = \{A\}$ and $t_{12} = 1$. $\mathcal{T}_{13} = \{E\}$ and $t_{13} = 1$.
$\mathcal{T}_{23} = \{F,G,H,I\}$ and $t_{23} = 4$.

Solution (c) for Condition 2 when $m = 5$



Treatment A occurs on all pairs involving individual 1.
Each other treatment is involved with each other individual exactly once.

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Some combinatorialists say that Condition 2 is satisfied
if the treatments give an equitable partition of the graph.