Designs on strongly regular graphs

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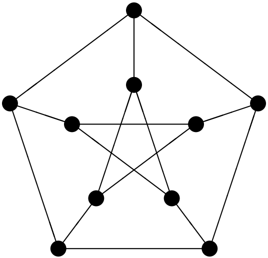
every vertex is contained in d edges.

The graph Γ is strongly regular if

- it is regular;
- if two vertices are joined by an edge, then they have p common neighbours, for some constant p;
- if two vertices are not joined by an edge, then they have q common neighbours, for some constant q;
- the graph is neither complete nor null.

The Petersen graph

This is a famous strongly regular graph.



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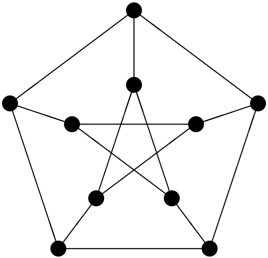

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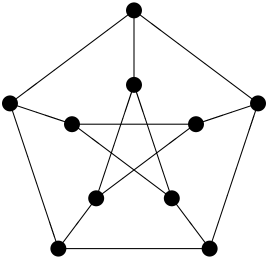

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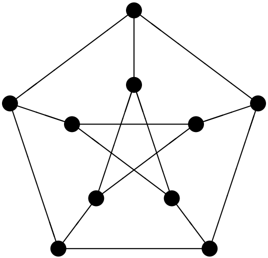

It has 10 vertices, each having degree 3.
If two vertices are joined by an edge, then they have no common neighbours.
If two vertices are not joined by an edge, then they have exactly one common neighbour.

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Here are three $\Omega \times \Omega$ real matrices associated with Γ:

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In this case, the real vector space $\mathbb{R}^\Omega$ is the orthogonal direct sum of subspaces W_0, W_1 and W_2, each of which is (contained in) an eigenspace of A and an eigenspace of J, where W_0 is the one-dimensional subspace spanned by the all-1 vector $\mathbf{u}$.

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(I will identify W_1 and W_2 later, as these depend on Γ.)

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I will describe two different desirable statistical conditions that translate easily into combinatorics and linear algebra.

I will illustrate each of these conditions when applied to the same two combinatorial objects (aka networks).

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When is the choice of best design not affected by the values of γ_0, γ_1 and γ_2?

Two different desirable statistical conditions

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Condition 2: We want the linear combination of the Y_ω (for $\omega \in \Omega$) which gives the best estimate of $\tau_C - \tau_D$ (correct on average, smallest variance) to be the same as the best estimator when $\gamma_0 = \gamma_1 = \gamma_2$. This is the difference between the averages for vertices with treatment C and those with treatment D.

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Solution The subspace V_T of $\mathbb{R}^\Omega$ consisting of vectors which are constant on each treatment can be orthogonally decomposed as

$$W_0 \oplus (V_T \cap W_1) \oplus (V_T \cap W_2).$$

Combinatorial Structure 1: Partition into Blocks

This is probably the best-known combinatorial structure in Design of Experiments.
The set Ω is partitioned into b blocks, each of size k.
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If $k = t$ then each block must contain every treatment.
If $k > t$ then something slightly more complicated is needed.

An example of a balanced incomplete-block design

Here is a balanced incomplete-block design with $b = 14$, $k = 4$, $t = 8$ and $\lambda = 3$.

1	3	5	7	2	4	6	8
1	2	5	6	3	4	7	8
1	2	3	4	5	6	7	8
1	4	5	8	2	3	6	7
1	3	6	8	2	4	5	7
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			2	3	4	2	3	5	2	4	5

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I don't want to get bogged down in the statistical details, so I will say no more about this here.

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Since the treatment subspace V_T contains W_0, there are three possibilities.

(a) $V_T \le W_0 \oplus W_2$.

(b) $V_T \le W_0 \oplus W_1$.

(c) $V_T \cap W_1$ and $V_T \cap W_2$ are both non-zero, and $V_T = W_0 \oplus (V_T \cap W_1) \oplus (V_T \cap W_2)$.

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More generally, any subset of treatments may be merged into a single treatment. For example,

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Such designs are used when management constraints make it impractical to apply the treatments to the individual plots.

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We combine the two previous approaches.
The treatment set is $\mathcal{T}_1 \times \mathcal{T}_2$,
where $|\mathcal{T}_1| = t_1$, which divides b, and $|\mathcal{T}_2| = k$.
Each item from $\mathcal{T}_2$ is applied to one plot per block.
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For example, when $b = 4$, $k = 3$, $t = 6$ and $t_1 = 2$ we get

$A1$	$A2$	$A3$	$B1$	$B2$	$B3$	$A1$	$A2$	$A3$	$B1$	$B2$	$B3$

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These are called split-plot designs,
and are widely used in practice.

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For example, the aim of the experiment might be to compare different methods for researchers to collaborate when they are unable to meet face-to-face, such as email, online meetings, old-fashioned letters, telephone calls with and without video.

Combinatorial Structure 2: more detail

Now the set Ω consists of all unordered pairs from the set $\{1, 2, \ldots, m\}$ of m distinct individuals, where $m \geq 4$.

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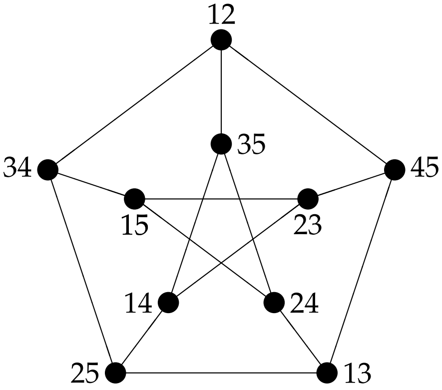
This is called the triangular graph $T(m)$.
It is strongly regular, and its adjacency matrix A satisfies

$$A^2 = (2m-8)I + (m-6)A + 4J.$$

The Petersen graph again

This labelling of the vertices shows that it is the complement of the triangular graph $T(5)$.



How to picture the vertices of $T(m)$ in general

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$$* = \{3, 5\}$$

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	1	2	3	4	5
2					
3	$\circ$	$\circ$			
4			$\circ$		
5	$\circ$	$\circ$	$*$	$\circ$	
6			$\circ$		$\circ$

$$* = \{3, 5\}$$

$\circ$ = vertices joined to vertex $\{3, 5\}$

Triangular graph: Condition 1

Condition 1 We want the variance V_{CD} of the estimator of $\tau_C - \tau_D$ to be the same for all pairs $\{C, D\}$ of distinct treatments.

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If m is odd and $t = m$ we can do this by using a symmetric, idempotent Latin square of order m and omitting the main diagonal and plots above the main diagonal (idempotent means that this diagonal contains each letter once).
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(Use the Cayley table of any Abelian group of odd order m.)
Then each treatment occurs on $(m-1)/2$ plots, and $\lambda = m - 2$.
In fact, each treatment misses one individual and occurs once with every other individual.

An example with $m = 7$

	1	2	3	4	5	6
2	B					
3	C	D				
4	D	E	F			
5	E	F	G	A		
6	F	G	A	B	C	
7	G	A	B	C	D	E

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Treatment A occurs once with every individual except individual 1.

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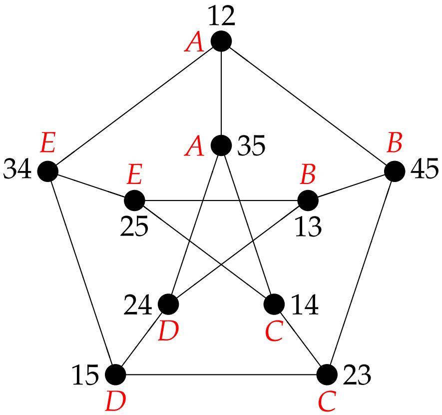
For strongly regular graphs in general, such designs are called balanced colourings of strongly regular graphs.

This design on the Petersen graph



For each treatment, there is one edge that has that treatment on both vertices.
For each pair of distinct treatments, there is one edge that has them on its endpoints.

Triangular Graph: the other Condition

For $i = 1, \ldots, m$, let $\mathbf{v}_i$ be the vector taking the value 1 on each pair that includes individual i and value 0 elsewhere. Let V_{ind} be the m-dimensional subspace of $\mathbb{R}^\Omega$ spanned by $\mathbf{v}_1, \ldots, \mathbf{v}_m$.

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Since the treatment subspace V_T contains W_0, there are three possibilities.

(a) $V_T \leq W_0 \oplus W_2$.

(b) $V_T \leq W_0 \oplus W_1$.

(c) $V_T \cap W_1$ and $V_T \cap W_2$ are both non-zero, and $V_T = W_0 \oplus (V_T \cap W_1) \oplus (V_T \cap W_2)$.

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In this case, we can do this by using a symmetric Latin square of order m with a single letter on the main diagonal and omitting the main diagonal and plots above the main diagonal.
(Start with a Latin square of the previous type;
add an extra row at the bottom;
move every diagonal element down to the bottom row;
then put a dummy like ∞ on every diagonal cell.)

An example with $m = 8$

	1	2	3	4	5	6	7
2	C						
3	D	E					
4	E	F	G				
5	F	G	A	B			
6	G	A	B	C	D		
7	A	B	C	D	E	F	
8	B	D	F	A	C	E	G

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Each treatment occurs exactly once with each individual.

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Just as with complete-block designs, any subset of treatments may be merged into a single treatment.

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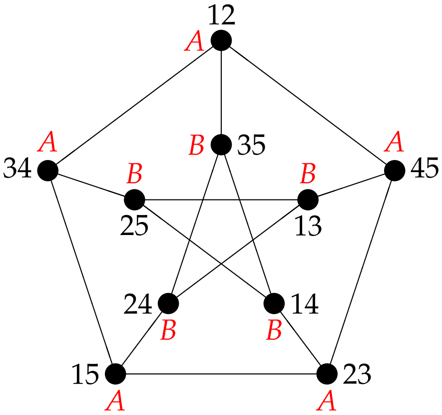
When $m = 9$ this gives

	1	2	3	4	5	6	7	8
2	1							
3	2	1						
4	3	2	1					
5	4	3	2	1				
6	4	4	3	2	1			
7	3	4	4	3	2	1		
8	2	3	4	4	3	2	1	
9	1	2	3	4	4	3	2	1

Solution (a) for Condition 2 when $m = 5$



Here A represents $\pm 1 \bmod 5$ and B represents $\pm 2 \bmod 5$.

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There are precisely two treatments, say A and B. There is one special individual i. Treatment A is applied to all pairs containing i, and treatment B is applied to all other pairs.

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9	A	B	B	B	B	B	B	B

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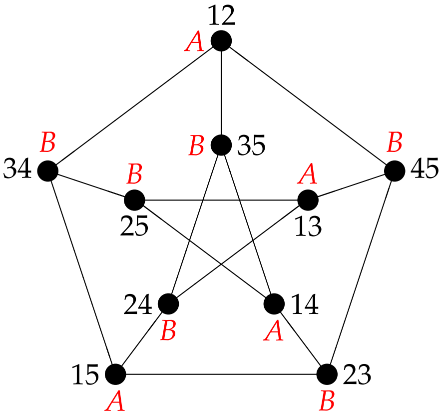

The two treatments are not equally replicated.

Solution (c) for Condition 2

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- If $s_i = 3$ then the only way to avoid replication 1 is to have $t_i = 1$.

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- If $s_i = 3$ then the only way to avoid replication 1 is to have $t_i = 1$.
- If $n = 2$ and $s_1 = 1$ then make sure that $t_2 > 1$, to avoid solution (b).

Solution (c) for Condition 2

(c) $V_T \cap W_1$ and $V_T \cap W_2$ are both non-zero, and $V_T = W_0 \oplus (V_T \cap W_1) \oplus (V_T \cap W_2)$.
Here is a very general solution.

- Partition the set of individuals into n sorts $\mathcal{S}_1, \ldots, \mathcal{S}_n$ of size $s_1, \ldots, s_n$, where $n \geq 2$.
- If $s_i > 1$ then put a solution (a) design on pairs of individuals of sort i, using t_i treatments forming a set $\mathcal{T}_i$.
- If $s_i = 2$ then $\mathcal{T}_i$ has a single treatment with replication 1, so avoid this case.
- If $s_i = 3$ then the only way to avoid replication 1 is to have $t_i = 1$.
- If $n = 2$ and $s_1 = 1$ then make sure that $t_2 > 1$, to avoid solution (b).
- If $i < j$ then let t_{ij} be any common divisor of s_i and s_j. Make a set $\mathcal{T}_{ij}$ of t_{ij} treatments. Allocate these to the cells in the rectangle $\mathcal{S}_j \times \mathcal{S}_i$ in such a way that all treatments appear equally often in each row and equally often in each column.

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- If $i < j$ and $s_i = s_j = 1$ then $\mathcal{T}_{ij}$ has a single treatment with replication 1, so avoid this case.

Theorem about this solution

Theorem

For $i = 1, \ldots, n$,
let $\mathbf{w}_i$ be the vector whose entries are

$$\begin{cases} 0 & \textit{on all pairs which do not involve an individual of sort } i \\ 1 & \textit{on all pairs which involve a single individual of sort } i \\ 2 & \textit{on all pairs which involve two indiviudals of sort } i \end{cases}$$

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Then

- *The vectors $\mathbf{w}_1, \ldots, \mathbf{w}_n$ span an n-dimensional subspace of $V_T \cap (W_0 \oplus W_1)$.*
- *If $\mathbf{v} \in V_T$ is orthogonal to $\mathbf{w}_i$ for $i = 1, \ldots, n$ then $\mathbf{v} \in W_2$.*

An example with two sorts

Here $m = 9$, $n = 2$, $s_1 = 3$, $s_2 = 6$ and $t = 9$.

	1	2	3	4	5	6	7	8
2	A							
3	A	A						
4	B	C	D					
5	B	C	D	E				
6	D	B	C	F	I			
7	D	B	C	G	H	E		
8	C	D	B	H	F	G	I	
9	C	D	B	I	G	H	F	E

An example with two sorts

Here $m = 9$, $n = 2$, $s_1 = 3$, $s_2 = 6$ and $t = 9$.

	1	2	3	4	5	6	7	8
2	A							
3	A	A						
4	B	C	D					
5	B	C	D	E				
6	D	B	C	F	I			
7	D	B	C	G	H	E		
8	C	D	B	H	F	G	I	
9	C	D	B	I	G	H	F	E

$\mathcal{S}_1 = \{1, 2, 3\}$, $\mathcal{T}_1 = \{A\}$ and $t_1 = 1$.

An example with two sorts

Here $m = 9$, $n = 2$, $s_1 = 3$, $s_2 = 6$ and $t = 9$.

	1	2	3	4	5	6	7	8
2	A							
3	A	A						
4	B	C	D					
5	B	C	D	E				
6	D	B	C	F	I			
7	D	B	C	G	H	E		
8	C	D	B	H	F	G	I	
9	C	D	B	I	G	H	F	E

$\mathcal{S}_1 = \{1, 2, 3\}$, $\mathcal{T}_1 = \{A\}$ and $t_1 = 1$.
$\mathcal{S}_2 = \{4, 5, 6, 7, 8, 9\}$, $\mathcal{T}_2 = \{E, F, G, H, I\}$ and $t_2 = 5$.

An example with two sorts

Here $m = 9$, $n = 2$, $s_1 = 3$, $s_2 = 6$ and $t = 9$.

	1	2	3	4	5	6	7	8
2	A							
3	A	A						
4	B	C	D					
5	B	C	D	E				
6	D	B	C	F	I			
7	D	B	C	G	H	E		
8	C	D	B	H	F	G	I	
9	C	D	B	I	G	H	F	E

$\mathcal{S}_1 = \{1, 2, 3\}$, $\mathcal{T}_1 = \{A\}$ and $t_1 = 1$.
$\mathcal{S}_2 = \{4, 5, 6, 7, 8, 9\}$, $\mathcal{T}_2 = \{E, F, G, H, I\}$ and $t_2 = 5$.
$\mathcal{T}_{12} = \{B, C, D\}$ and $t_{12} = 3$.

An example with three sorts

Here $m = 9$, $n = 3$, $s_1 = 1$, $s_2 = 4$, $s_3 = 4$ and $t = 12$.

	1	2	3	4	5	6	7	8
2	A							
3	A	B						
4	A	C	D					
5	A	D	C	B				
6	E	F	G	H	I			
7	E	G	H	I	F	J		
8	E	H	I	F	G	K	L	
9	E	I	F	G	H	L	K	J

An example with three sorts

Here $m = 9$, $n = 3$, $s_1 = 1$, $s_2 = 4$, $s_3 = 4$ and $t = 12$.

	1	2	3	4	5	6	7	8
2	A							
3	A	B						
4	A	C	D					
5	A	D	C	B				
6	E	F	G	H	I			
7	E	G	H	I	F	J		
8	E	H	I	F	G	K	L	
9	E	I	F	G	H	L	K	J

$\mathcal{S}_1 = \{1\}$, $\mathcal{T}_1 = \emptyset$ and $t_1 = 0$.

An example with three sorts

Here $m = 9$, $n = 3$, $s_1 = 1$, $s_2 = 4$, $s_3 = 4$ and $t = 12$.

	1	2	3	4	5	6	7	8
2	A							
3	A	B						
4	A	C	D					
5	A	D	C	B				
6	E	F	G	H	I			
7	E	G	H	I	F	J		
8	E	H	I	F	G	K	L	
9	E	I	F	G	H	L	K	J

$\mathcal{S}_1 = \{1\}$, $\mathcal{T}_1 = \emptyset$ and $t_1 = 0$.
$\mathcal{S}_2 = \{2, 3, 4, 5\}$, $\mathcal{T}_2 = \{B, C, D\}$ and $t_2 = 3$.

An example with three sorts

Here $m = 9$, $n = 3$, $s_1 = 1$, $s_2 = 4$, $s_3 = 4$ and $t = 12$.

	1	2	3	4	5	6	7	8
2	A							
3	A	B						
4	A	C	D					
5	A	D	C	B				
6	E	F	G	H	I			
7	E	G	H	I	F	J		
8	E	H	I	F	G	K	L	
9	E	I	F	G	H	L	K	J

$\mathcal{S}_1 = \{1\}$, $\mathcal{T}_1 = \emptyset$ and $t_1 = 0$.
$\mathcal{S}_2 = \{2,3,4,5\}$, $\mathcal{T}_2 = \{B,C,D\}$ and $t_2 = 3$.
$\mathcal{S}_3 = \{6,7,8,9\}$, $\mathcal{T}_3 = \{J,K,L\}$ and $t_3 = 3$.

An example with three sorts

Here $m = 9$, $n = 3$, $s_1 = 1$, $s_2 = 4$, $s_3 = 4$ and $t = 12$.

	1	2	3	4	5	6	7	8
2	A							
3	A	B						
4	A	C	D					
5	A	D	C	B				
6	E	F	G	H	I			
7	E	G	H	I	F	J		
8	E	H	I	F	G	K	L	
9	E	I	F	G	H	L	K	J

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$\mathcal{T}_{12} = \{A\}$ and $t_{12} = 1$.

An example with three sorts

Here $m = 9$, $n = 3$, $s_1 = 1$, $s_2 = 4$, $s_3 = 4$ and $t = 12$.

	1	2	3	4	5	6	7	8
2	*A*							
3	*A*	*B*						
4	*A*	*C*	*D*					
5	*A*	*D*	*C*	*B*				
6	*E*	*F*	*G*	*H*	*I*			
7	*E*	*G*	*H*	*I*	*F*	*J*		
8	*E*	*H*	*I*	*F*	*G*	*K*	*L*	
9	*E*	*I*	*F*	*G*	*H*	*L*	*K*	*J*

$\mathcal{S}_1 = \{1\}$, $\mathcal{T}_1 = \emptyset$ and $t_1 = 0$.
$\mathcal{S}_2 = \{2,3,4,5\}$, $\mathcal{T}_2 = \{B,C,D\}$ and $t_2 = 3$.
$\mathcal{S}_3 = \{6,7,8,9\}$, $\mathcal{T}_3 = \{J,K,L\}$ and $t_3 = 3$.
$\mathcal{T}_{12} = \{A\}$ and $t_{12} = 1$. $\mathcal{T}_{13} = \{E\}$ and $t_{13} = 1$.

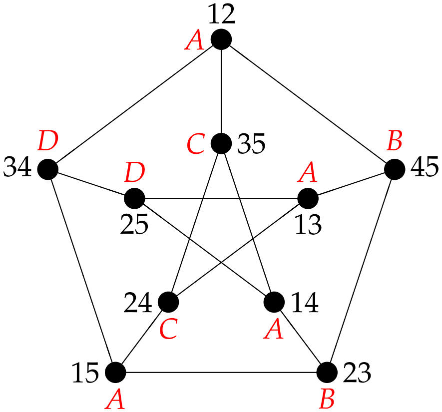
An example with three sorts

Here $m = 9$, $n = 3$, $s_1 = 1$, $s_2 = 4$, $s_3 = 4$ and $t = 12$.

	1	2	3	4	5	6	7	8
2	A							
3	A	B						
4	A	C	D					
5	A	D	C	B				
6	E	F	G	H	I			
7	E	G	H	I	F	J		
8	E	H	I	F	G	K	L	
9	E	I	F	G	H	L	K	J

$\mathcal{S}_1 = \{1\}$, $\mathcal{T}_1 = \emptyset$ and $t_1 = 0$.
$\mathcal{S}_2 = \{2,3,4,5\}$, $\mathcal{T}_2 = \{B,C,D\}$ and $t_2 = 3$.
$\mathcal{S}_3 = \{6,7,8,9\}$, $\mathcal{T}_3 = \{J,K,L\}$ and $t_3 = 3$.
$\mathcal{T}_{12} = \{A\}$ and $t_{12} = 1$. $\mathcal{T}_{13} = \{E\}$ and $t_{13} = 1$.
$\mathcal{T}_{23} = \{F,G,H,I\}$ and $t_{23} = 4$.

Solution (c) for Condition 2 when $m = 5$



Treatment A occurs on all pairs involving individual 1.
Each other treatment is involved with each other individual exactly once.

Terminology

For a wide range of structures on the set Ω,
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Some combinatorialists say that Condition 2 is satisfied
if the treatments give an equitable partition of the graph.

References

- R. A. Bailey:
  Balanced colourings of strongly regular graphs.
  Discrete Mathematics **293** (2005), 73–90.
  doi: 10.1016/j.disc.2004.08.022

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- R. A. Bailey, Peter J. Cameron, Dário Ferreira,
  Sandra S. Ferreira and Célia Nunes:
  Designs for half-diallel experiments with commutative orthogonal block structure.
  Journal of Statistical Planning and Inference, **231** (2024), 106139. doi: 10.1016/j.jspi.2023.106139