Semi-Latin Squares and their Extensions

R. A. Bailey
University of St Andrews

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What is a semi-Latin Square?

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In the case that $k = 1$, we just get a Latin square of order n,
so we always assume that $k > 1$.

An example of a semi-Latin square

$A \quad \alpha \quad a$	$B \quad \beta \quad b$	$C \quad \gamma \quad c$	$D \quad \delta \quad d$	$E \quad \varepsilon \quad e$
$E \quad \delta \quad c$	$A \quad \varepsilon \quad d$	$B \quad \alpha \quad e$	$C \quad \beta \quad a$	$D \quad \gamma \quad b$
$D \quad \beta \quad e$	$E \quad \gamma \quad a$	$A \quad \delta \quad b$	$B \quad \varepsilon \quad c$	$C \quad \alpha \quad d$
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A α a	B β b	C γ c	D δ d	E ε e
E δ c	A ε d	B α e	C β a	D γ b
D β e	E γ a	A δ b	B ε c	C α d
C ε b	D α c	E β d	A γ e	B δ a
B γ d	C δ e	D ε a	E α b	A β c

Here there are 5 rows, 5 columns, and 3 plots per cell.

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There are 15 treatments,
each occurring once in each row and once in each column.

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Once people started to include the cells in the data analysis, Frank Yates became enthusiastic about using semi-Latin squares.

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This randomization justifies the linear model whose variance-covariance matrix has the following eigenspaces (also called strata).

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Rows	$n-1$
Columns	$n-1$
Cells within Rows and Columns	$(n-1)^2$
Plots within Cells	$n^2(k-1)$

General balance

Treatment contrasts are orthogonal to the first three strata,
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This property was called general balance by Nelder (1965).

Trojan squares

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Our previous example with $n = 5$ and $k = 3$ is Trojan.

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E δ c	A ε d	B α e	C β a	D γ b
D β e	E γ a	A δ b	B ε c	C α d
C ε b	D α c	E β d	A γ e	B δ a
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Theorem

If there is a Trojan square of size $(n \times n)/(n-1)$, then all of its inflations are A-optimal.

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Soicher proved that there exists a $(6 \times 6)/5s$ uniform semi-Latin square whenever $s \geq 2$.

Semi-Latin rectangles

The first way that I extend the notion of semi-Latin squares is by relaxing the condition that each treatment occurs precisely once in each row and once in each column. We maintain the rules that

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RAB and Hervé Monod first did this in 2001 for an experiment where the experimental units were the half-leaves. The p plants can be thought of as columns, and the h leaf-heights can be thought of as rows. In this case, $k = 2$.

An example from an experiment on plant disease

height 4	D	E	E	F	F	G	G	H	H	A	A	B	B	C	C	D
height 3	A	G	B	H	C	A	D	B	E	C	F	D	G	E	H	F
height 2	H	C	A	D	B	E	C	F	D	G	E	H	F	A	G	B
height 1	B	F	C	G	D	H	E	A	F	B	G	C	H	D	A	E
	plant 1		plant 2		plant 3		plant 4		plant 5		plant 6		plant 7		plant 8	

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height 1	B	F	C	G	D	H	E	A	F	B	G	C	H	D	A	E

The treatment pairs $\{A,E\}$, $\{B,F\}$, $\{C,G\}$ and $\{D,H\}$, each come in two different cells in the bottom row.

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So this design is A-optimal.

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So if each treatment occurs s times in each row and s times in each column, we have to call it a semi-Latin rectangle if $s \geq 2$.

A balanced semi-Latin rectangle

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2 3 5	3 4 6	4 5 7	5 6 1	6 7 2	7 1 3	1 2 4
3 4 6	4 5 7	5 6 1	6 7 2	7 1 3	1 2 4	2 3 5
4 5 7	5 6 1	6 7 2	7 1 3	1 2 4	2 3 5	3 4 6
5 6 1	6 7 2	7 1 3	1 2 4	2 3 5	3 4 6	4 5 7
6 7 2	7 1 3	1 2 4	2 3 5	3 4 6	4 5 7	5 6 1
7 1 3	1 2 4	2 3 5	3 4 6	4 5 7	5 6 1	6 7 2

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Because North–South variation is usually greater per distance than East–West variation, statisticians in agricultural research in the UK decided in the 1980s that it is reasonable to divide the glasshouse into n rows and n columns,

where each row is an East–West line of nk plots and

each column consists of k contiguous North–South lines of plots.

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This view led to widespread use of semi-Latin squares in agricultural research.

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Now the n^2 cells of size k and the nk lines of size n both form systems of incomplete-blocks, so we have to allow for both of them in our design, randomization and data analysis.

Structure

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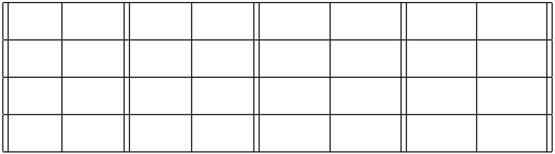

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There are 4 columns, each containing 2 lines of 4 plots.
Within each column, cells are crossed with lines.

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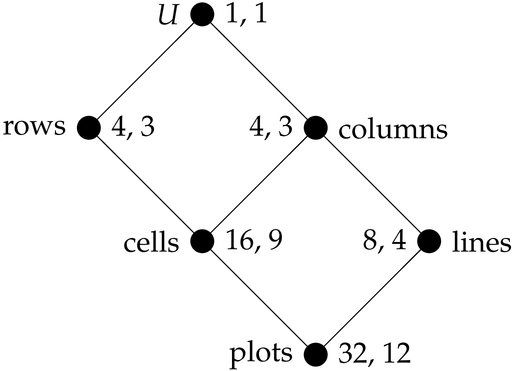
Plots within Cells and Lines	$n(n-1)(k-1)$

Hasse diagram of block factors in the example



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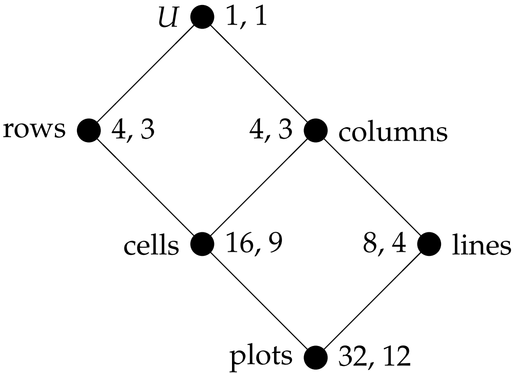

Here, U is the universal factor, which has a single level.
For each factor, the first number shows its number of levels and the second number shows its number of degrees of freedom.

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RAB's approach is to use patterns and combinatorics.
Since more information is lost to the block system with the smaller block size, it is better to start by finding the best design for that block system.

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stratum	dim	A	B	C	AB	AC	BC	ABC
cells	9	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
lines	4							
plots	12							

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cells	9	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
lines	4	$\frac{1}{2}$						
plots	12							

Next, consider the lines of size 4. There are two lines in each column. In any column, the factor A can be used to define the lines. If a factor is used like this in m columns, then its loss of information to lines is $m/4$. Let us use A in the first two columns.

An example of my approach

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Calculating the A-criterion in the example

stratum	dim	A	B	C	AB	AC	BC	ABC
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Then the A-criterion is

$$\left(\frac{5\times 2+2\times 4}{7}\right)^{-1}=\frac{7}{18}=0.3889.$$

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Then the A-criterion is

$$\left(\frac{5 \times 2 + 2 \times 4}{7}\right)^{-1} = \frac{7}{18} = 0.3889.$$

In fact, Emlyn's software shows that this design is A-optimal.

Unrandomized version of the design in that example

6	2	8	4	1	7	5	3
8	1	6	3	5	2	7	4
7	3	5	1	6	4	1	8
5	4	7	2	3	8	2	6

Translation:

$$1 = 1 \quad 2 = c \quad 3 = b \quad 4 = bc \quad 5 = a \quad 6 = ab \quad 7 = ac \quad 8 = abc$$

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This method works best when the matrices of projection onto all three strata have the same eigenvectors;
in other words, we have general balance.

References

- E. R. Williams and R. A. Bailey (2026) Extended semi-Latin squares for use in field and glasshouse trials. *Journal of Agricultural, Biological and Environmental Statistics*, doi: `10.1007/s13253-026-00729-6`
- N. P. Uto (2021) Construction and comparison of semi-Latin rectangles.
  PhD thesis, University of St Andrews.
- L. H. Soicher (2012) Uniform semi-Latin squares and their Schur-optimality. *Journal of Combiatorial Designs* **20**, 265–277.
- R. A. Bailey and H. Monod (2001) Efficient semi-Latin rectangles: designs for plant disease experiments. *Scandinavian Journal of Statistics* **28**, 257–270.
- J. A. Nelder (1965) The analysis of randomized experiments with orthogonal block structure. II. Treatment structure and the general analysis of variance. *Proceedings of the Royal Society of London, Series A* **283**, 163–178.