Poset block structures and generalized wreath products

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Joint work with Peter Cameron (University of St Andrews) and Marina Anagnostopoulou–Merkouri (University of Bristol)

Preliminaries

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By definition, because G is transitive, all of the partitions which it preserves are uniform.

An example: a Cayley table

The Cayley table of $(\mathbb{Z}_5, +)$ is this Latin square.

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$G = \mathbb{Z}_5 \times \mathbb{Z}_5$ acts on this as follows.
The permutation (g, h) sends the element (x, y) to $(g^{-1}x, yh)$.
(So that if $x_1y_1 = x_2y_2$ then $g^{-1}x_1y_1h = g^{-1}x_2y_2h$.)

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The group G preserves the three non-trivial partitions into Rows, Columns and "Letters", as well as E and U.

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This can be read "Π_1 refines Π_2" or "Π_2 is coarser than Π_1".
Given a collection $\mathcal{P}$ of partitions of a set Ω,
we can show them on a Hasse diagram.

- Draw a dot for each partition in $\mathcal{P}$.
- If $\Pi_1 \prec \Pi_2$ then put Π_2 higher than Π_1 in the diagram.
- If $\Pi_1 \prec \Pi_2$ but there is no Π_3 in $\mathcal{P}$ with $\Pi_1 \prec \Pi_3 \prec \Pi_2$ then draw a line from Π_1 to Π_2.

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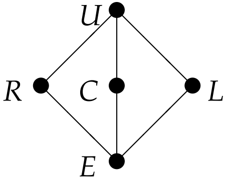
Here is the Hasse diagram for a Latin square.



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We say that partitions Π_1 and Π_2 commute with each other if, whenever α and β are in the same part of $\Pi_1 \vee \Pi_2$,
there exist γ and δ such that α and γ are in the same part of Π_1, γ and β are in the same part of Π_2, α and δ are in the same part of Π_2, and δ and β are in the same part of Π_1.

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In other words, we can move from α to β by first moving within a part of Π_1 then moving within a part of Π_2, and also do it by first moving within a part of Π_2 and then moving within a part of Π_1.

Orthogonal block structures

Here is an alternative definition of Latin square.

Definition
A Latin square is a set $\{R, C, L\}$ of pairwise commuting uniform partitions of a set Ω which satisfy $R \wedge C = R \wedge L = C \wedge L = E$ and $R \vee C = R \vee L = C \vee L = U$.

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So Latin squares are OBS.

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Iterated crossing and nesting gives simple orthogonal block structures.

Properties

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The lattice of partitions of a Latin square is modular but not distributive.

An OBS which is a distributive lattice is called a poset block structure or PBS.

Definition of Poset Block Structure

Let $(M, \sqsubseteq)$ be a partially ordered set.

Definition

A down-set in M is a subset D of M with the property that, if $m \in D$ and $m' \sqsubset m$, then $m' \in D$.

The down-sets form a distributive lattice under the operations of intersection and union. Every finite distributive lattice arises in this way.

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For each element m_i of M, let Ω_i be a set of size $n_i > 1$.

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Now we define a partition Π_D of Ω for each down-set D of M .
This is done as follows.

Definition

Elements $(\alpha_1, \ldots, \alpha_N)$ and $(\beta_1, \ldots, \beta_N)$ are in the same part of Π_D if and only if $\alpha_i = \beta_i$ for all i with $m_i \notin D$.

(The reason for this counter-intuitive definition becomes clear on the next slide.)

Easy consequences

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The statisticians who first invented this idea (in the 1950s) did make this mistake.
They tried to draw a single Hasse diagram showing both of these posets at once.

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In other words, if elements α and β in Ω agree on all
coordinates j which are ancestors of i,
then the same element of G_i acts on both α_i and β_i.

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We saw earlier that Latin squares give rise to OBSs (consisting of the two partitions E and U and the row, column, and letter partitions). It is known that almost all Latin squares have trivial automorphism group.

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3. If the poset $(M, \sqsubseteq)$ can be made by iterated crossing and nesting
   then the GWP can be made by iterating the corresponding direct and wreath products.
4. If the poset $(M, \sqsubseteq)$ cannot be made in this way,
   then neither can the GWP.

The Krasner–Kaloujnine theorem

The following result is well-known.

Theorem

Let G be a transitive imprimitive permutation group,
with a non-trivial invariant partition Π.
Then G is naturally embeddable in the wreath product $H \wr K$,
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Anyone who wants to know the details, please ask me for the extra slides.

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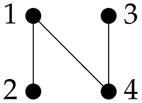
Theorem

Let G be a permutation group on Ω with the PB property, and let M be the corresponding poset and G_m^* the group defined above for each $m \in M$. Then G is naturally embedded in the generalized wreath product of the groups G_m^* over the poset M.

An example with $|M| = 4$



Put D_m to be the smallest down-set containing m.
Let Π_D be the partition defined by D.
This table shows the partitions as down-sets.

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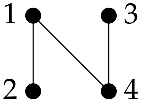

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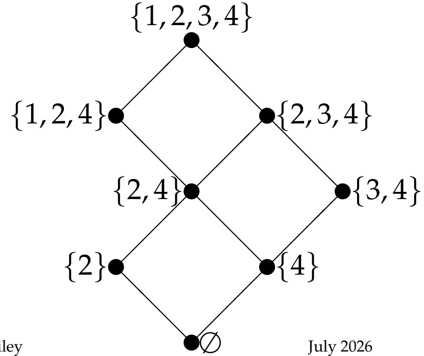

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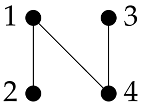

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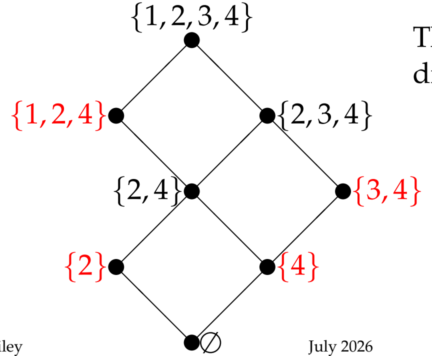

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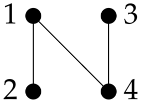

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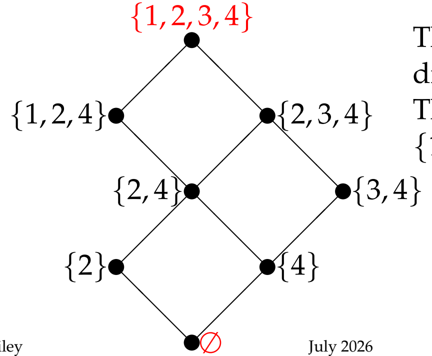

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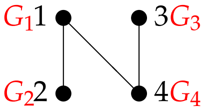

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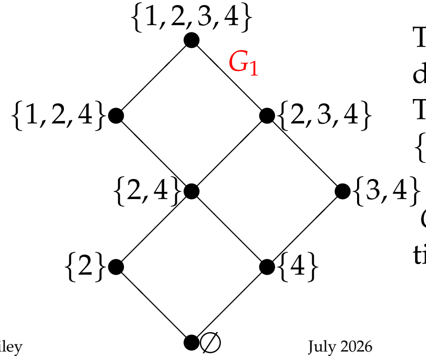

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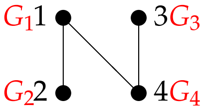

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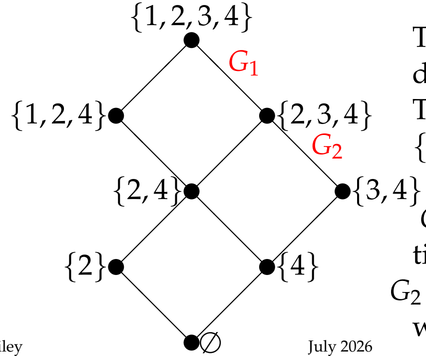

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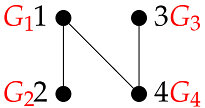

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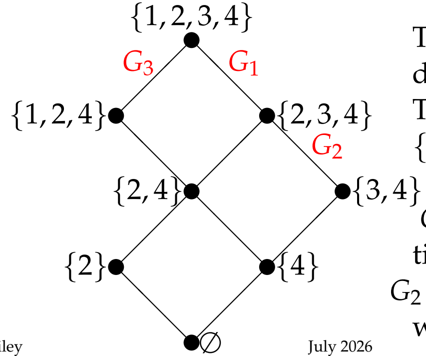

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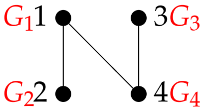

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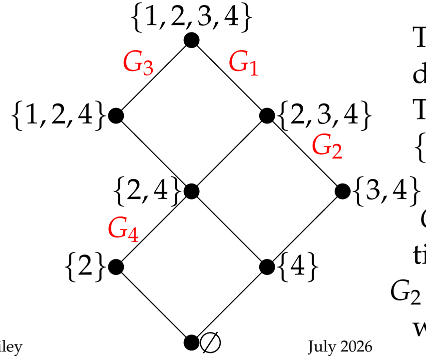

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