Extended semi-Latin squares

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Semi-Latin squares

A Latin square of order n is an $n \times n$ square array of n^2 cells, with n treatments, each occurring once in each row and once in each column.

A	B	C	D
B	A	D	C
C	D	A	B
D	C	B	A

Semi-Latin squares (my examples have $n = 4$ and $k = 2$)

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C	D	A	B
D	C	B	A

A E	B F	C G	D H
B G	A H	D E	C F
C H	D G	A F	B E
D F	C E	B H	A G

A semi-Latin square with parameters n and k is an $n \times n$ square array of n^2 cells, each containing k plots.
There are nk treatments, each occurring once in each row and once in each column.

Concurrence matrix

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We can show these in a $nk \times nk$ concurrence matrix Λ_1.

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C H	D G	A F	B E
D F	C E	B H	A G

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \\ G \\ H \end{array} \begin{array}{c} \begin{array}{cccccccc} A & B & C & D & E & F & G & H \end{array} \\ \left[\begin{array}{cccccccc} 4 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 4 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 4 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 4 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 4 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 4 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 4 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 4 \end{array}\right] \end{array}$$

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The eigenvectors and eigenvalues of Λ_1 determine important statistical properties of the design.

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B	G	A	H	E	D	C	F
C	H	D	G	F	A	E	B
D	F	C	E	B	H	G	A

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B	G	A	H	E	D	C	F
C	H	D	G	F	A	E	B
D	F	C	E	B	H	G	A

	A	B	C	D	E	F	G	H
A	4	3	2	3	0	1	1	2
B	3	4	3	2	1	2	0	1
C	2	3	4	3	2	1	1	0
D	3	2	3	4	1	0	2	1
E	0	1	2	1	4	3	3	2
F	1	2	1	0	3	4	2	3
G	1	0	1	2	3	2	4	3
H	2	1	0	1	2	3	3	4

Call its concurrence matrix Λ_2.

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This desirable property of a design is called general balance, a term introduced by John Nelder in 1965.

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Sometimes our approaches produce isomorphic designs; otherwise, his are better than mine
if we use the model that cells and lines have fixed effects.