From partitions to diagonal structures and beyond

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Outline

1. Partitions

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2. Some statistical history

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3. Diagonal semilattices

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4. Diagonal graphs

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5. … and beyond.

Chapter 1

Partitions

What is a Latin square?

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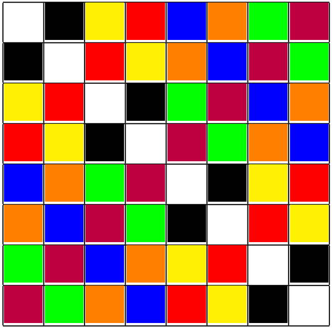
A Latin square of order 8



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Example

If Ω is the set of cells in a Latin square, then there are five natural uniform partitions of Ω:

- R each part is a row;
- C each part is a column;
- L each part consists of the those cells with a given letter;
- U the universal partition, with a single part;
- E the equality partition, whose parts are singletons.

The partial order on partitions of a set

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Draw a graph by putting an edge between two points if they
are in the same part of P or the same part of Q. Then
the parts of $P \vee Q$ are the connected components of the graph.

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Semigroup theorists call a semigroup satisfying these conditions a semilattice.

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The above conditions show that it is a special kind of semigroup.
Each such semigroup is isomorphic to one defined by a meet semilattice.

Hasse diagrams

Given a collection $\mathcal{P}$ of partitions of a set Ω,
we can show them on a Hasse diagram.

- Draw a dot for each partition in $\mathcal{P}$.
- If $P \prec Q$ then put Q higher than P in the diagram.
- If $P \prec Q$ but there is no S in $\mathcal{P}$ with $P \prec S \prec Q$ then draw a line from P to Q.

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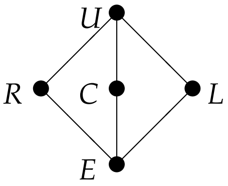
Here is the Hasse diagram for a Latin square.



An alternative definition of Latin square

Definition

Let P and Q be uniform partitions of a set Ω. Then P and Q are compatible if

- whenever ω_1 and ω_2 are points in the same part of $P \vee Q$, there are points α and β such that
  - ω_1 and α are in the same part of P,
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- $P \wedge Q$ is uniform.

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Definition

A Latin square is a set $\{R, C, L\}$ of pairwise compatible uniform partitions of a set Ω which satisfy $R \wedge C = R \wedge L = C \wedge L = E$ and $R \vee C = R \vee L = C \vee L = U$.

Another nice family of partitions

Definition

Suppose that P_1, P_2 and P_3 are partitions of a set Ω, none of which is U. Then $\{P_1, P_2, P_3\}$ is a Cartesian decomposition of Ω of dimension 3 if $|\Gamma_1 \cap \Gamma_2 \cap \Gamma_3| = 1$ whenever Γ_i is a part of P_i for $i = 1, 2, 3$.

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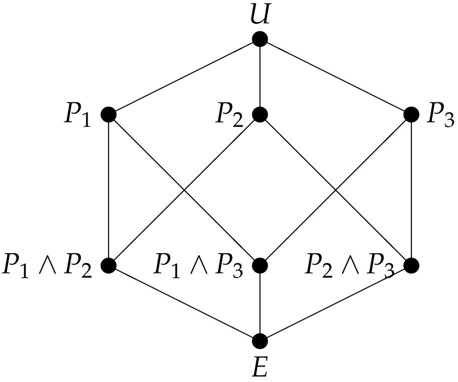

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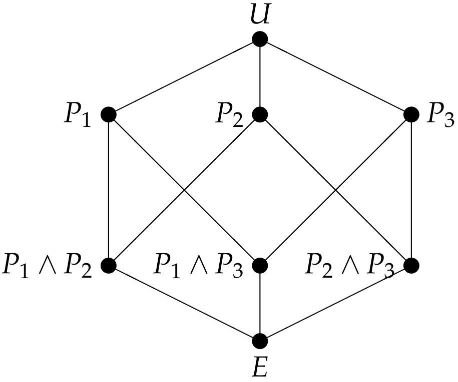

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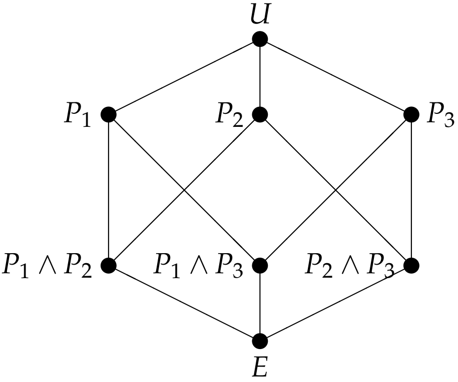

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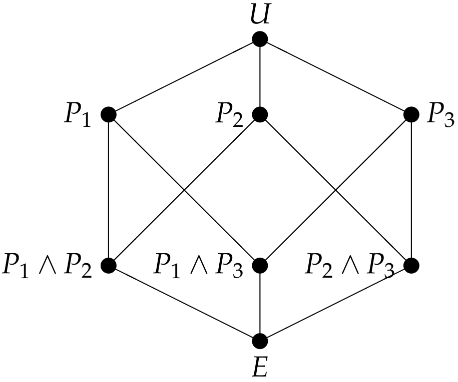

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- Statisticians call this a completely crossed orthogonal block structure.

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Proposition

Let H and K be subgroups of a group G. The following hold.

1. P_H *is uniform.*
2. $P_H \wedge P_K = P_{H \cap K}$.
3. $P_H \vee P_K = P_{\langle H,K \rangle}$.
4. P_H *and* P_K *are compatible if and only if* $HK = KH$.

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Theorem

If P and Q are uniform and compatible then
$V_P \cap V_{P \vee Q}^\perp$ is orthogonal to $V_Q \cap V_{P \vee Q}^\perp$.

Orthogonal decomposition

Theorem

Suppose that $\mathcal{P}$ is a join semilattice of pairwise compatible uniform partitions of Ω. For P in $\mathcal{P}$, put

$$W_P = V_P \cap \left(\sum_{P \prec Q} V_Q \right)^{\perp}.$$

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The partial order $\preccurlyeq$ has a zeta-function ζ defined by

$$\zeta(Q,P) = \begin{cases} 1 & \text{if } Q \preccurlyeq P, \\ 0 & \text{otherwise.} \end{cases}$$

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So $\dim(V_Q) = \sum_P \zeta(Q,P) \dim(W_P)$.

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Applying so-called Möbius inversion to the equation at the top of this slide gives

$$\dim(W_P) = \sum_Q \mu(P,Q)\dim(V_Q).$$

Chapter 2

Some statistical history

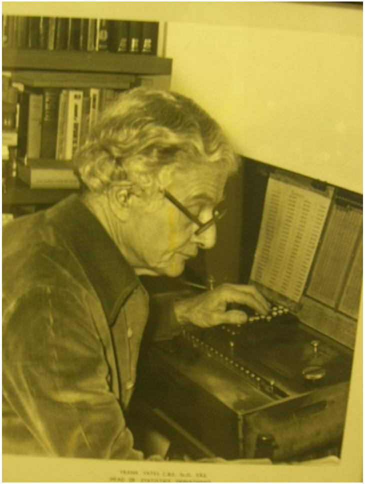
Statisticians at Rothamsted

Here are some of the statisticians who have worked at the agricultural research station at Rothamsted.

Ronald Fisher	1919–1933	then UCL, then Cambridge
Frank Yates	1931–1968	
David Finney	1939–1945	then Oxford, Aberdeen, then Edinburgh
Oscar Kempthorne	1941–1946	then Ames, Iowa
Desmond Patterson	1947–1967	then Edinburgh
John Nelder	1968–1984	previously National Vegetable Research Station
Rosemary Bailey	1981–1990	
Robin Thompson	1997–2012 (?)	previously Edinburgh

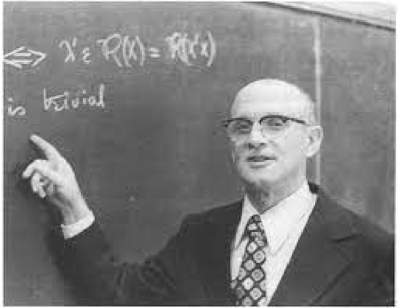
Photos: Fisher and Yates

Ronald Fisher



Frank Yates

Photos: Kempthorne and Patterson



Oscar Kempthorne

Desmond Patterson

Photos: Finney and Nelder

David Finney

John Nelder

Nelder's papers

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Desmond responded “Hmph! That’s good. No one else does.”

I did not believe him then, but, looking back, I can see that his approach did not incorporate Nelder’s ideas until much later.

Kempthorne's papers

Then my colleague Robin Thompson gave me a 1961 technical report (long, but in typescript) by Oscar Kempthorne and his colleagues in Ames. This developed essentially the same ideas as Nelder's: lattices of partitions using some of the partitions in a Cartesian lattice (not necessarily with all coordinates having the same number of values, for example, the rows and columns of a rectangle).

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Later I learnt that Kempthorne was furious that Nelder had "stolen" his ideas. I believe that they simply developed them independently, building on the work of Fisher and Yates. In those days, it took much longer for ideas to circulate widely.

Putting the bits together

One morning, I came into work after drinking too much in the pub the previous evening. I realised that my brain was not capable of serious work, so I gave it the apparently simple task of matching Nelder's block structures with those of Kempthorne. Slowly, I worked through dimensions 1, 2 and 3.

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The next day, with a clear head, I realised that Kempthorne's approach always gives more possibilities than Nelder's in dimensions at least 4.

Meeting Kempthorne

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In June 1988 I attended a two-week research workshop at the Institute for Mathematics and its Applications in Minneapolis, USA. At the weekend, another participant, Jonathan Smith, took me to Ames, so that I could have some meetings with Kempthorne. Kempthorne was very friendly, and said that he much appreciated my work, but

> *"This Möbius function really does the job. I wish that we had known about it."*

Chapter 3

Diagonal semilattices

Starting work on diagonal structures

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We started to collaborate, and two years later
(during the first Covid-19 lockdown) proved a lovely theorem.

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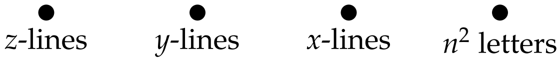

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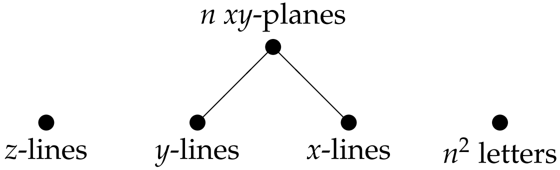

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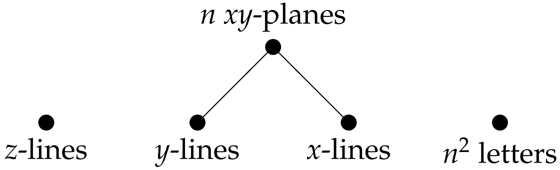

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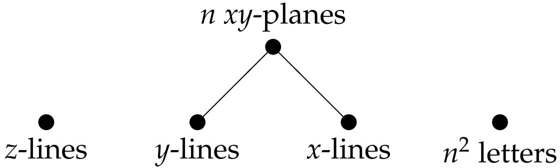

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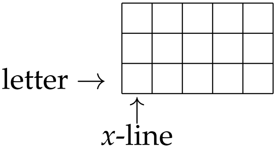

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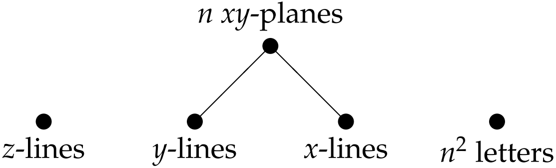

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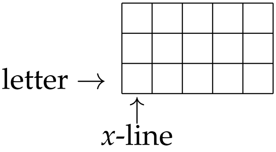

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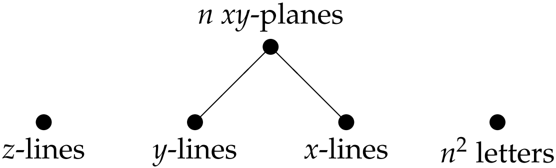

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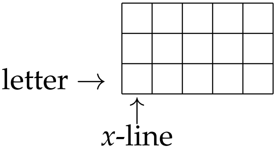

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Conditions (1) and (2) give one definition (among very many) of a Latin cube.

Some more statistics

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Put $x = ab^{-1}, y = bc^{-1}, z = cd^{-1}$ and $t = xyz = ad^{-1}$.
Then $H = \langle x \rangle \times \langle y \rangle \times \langle z \rangle$ and the coset partitions of H defined by any 3 of $\langle x \rangle$, $\langle y \rangle$, $\langle z \rangle$ and $\langle t \rangle$ are the minimal non-trivial partitions in a Cartesian lattice of dimension 3.

Theorem about diagonal semilattices

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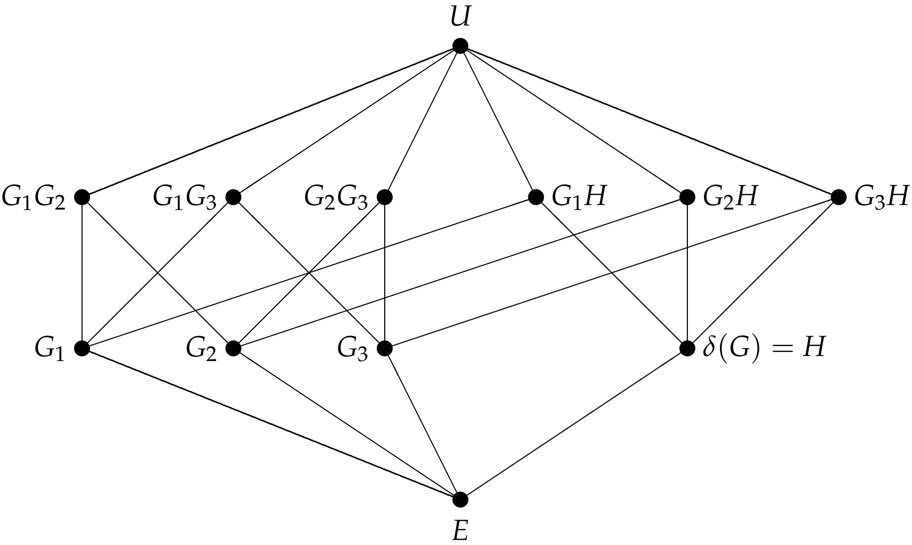
For $m > 2$, the combinatorial assumptions in the statement of the theorem force the existence of a group.

Hasse diagram for coset partitions in dimension 3



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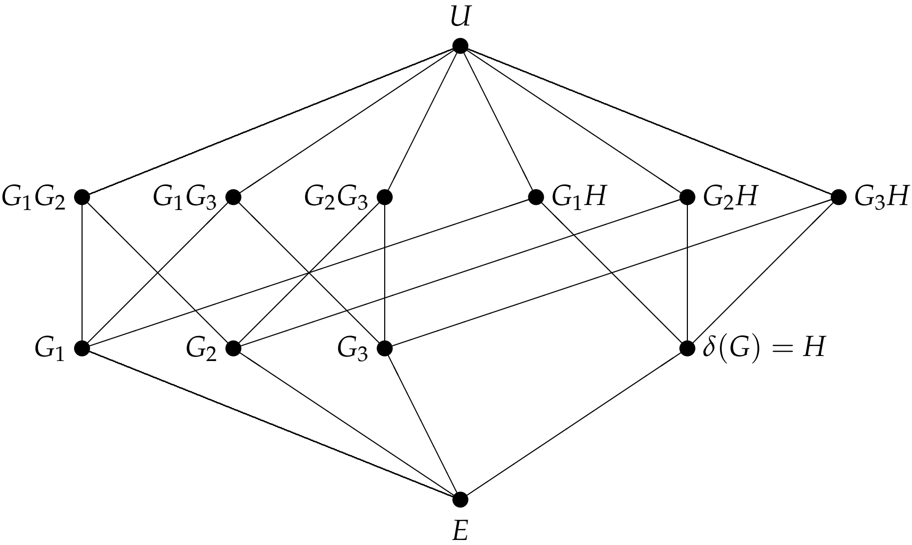

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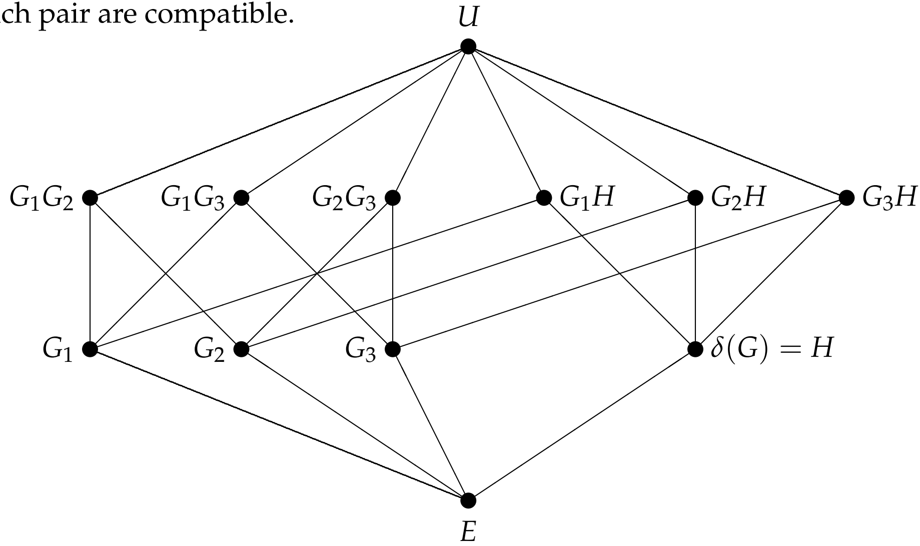

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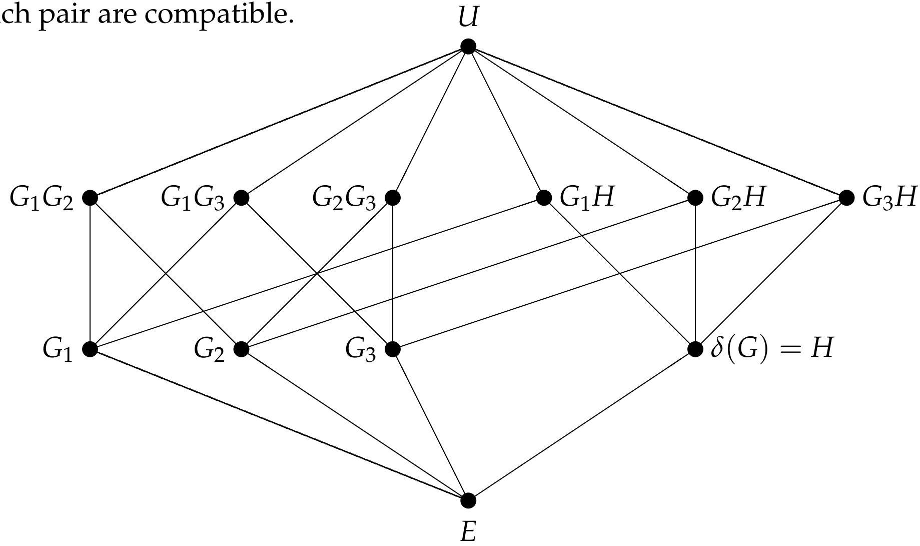

Hasse diagram for coset partitions in dimension 3

Each partition is uniform.

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All suprema are included,

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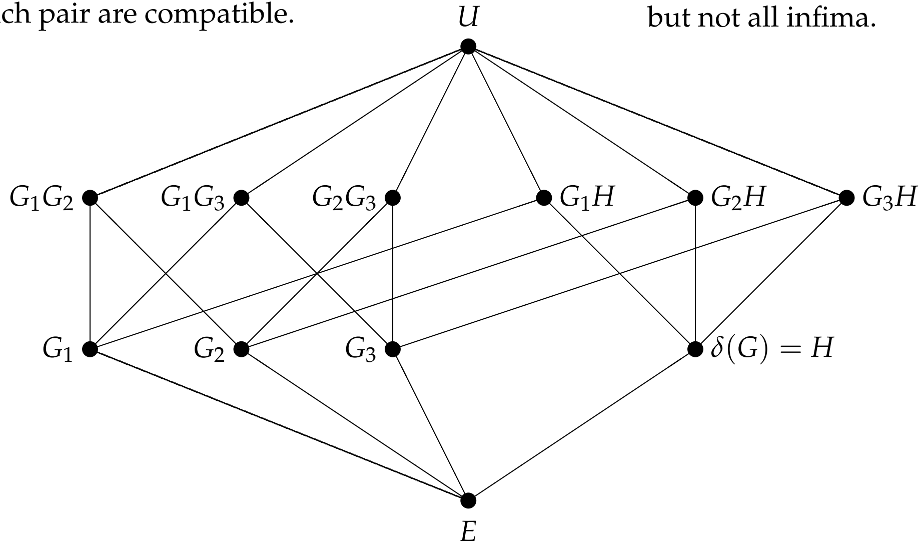

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2. In 1984, Danish statistician Tue Tjur pointed out that, for statistical purposes, closure under suprema is more important than closure under infima,
   and that such closure does not destroy compatibility.

Chapter 4

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In recent work, Peter Cameron and I have generalized the folded cube to larger values of n, using a diagonal semi-lattice.

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If $m = 2$, this is the Latin-square graph defined by the Cayley table of G. This is a well-known strongly regular graph.

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- The diameter is equal to

$$m+1-\left\lceil \frac{m+1}{n} \right\rceil,$$

  which is less than or equal to m,
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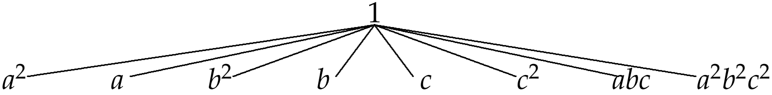
Here are the vertices joined to vertex 1.



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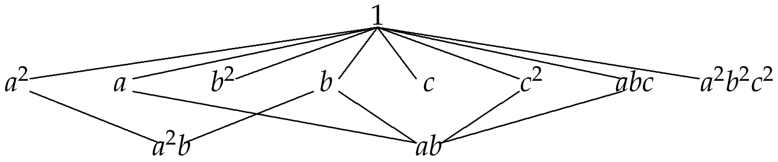
minimal partitions	Q_0	Q_1	Q_2	Q_3
cosets of	$\langle abc \rangle$	$\langle a \rangle$	$\langle b \rangle$	$\langle c \rangle$

Here are the vertices joined to vertex 1.



Vertices 1 and ab are at distance 2, and have 4 common neighbours. Vertices 1 and a^2b are at distance 2, and have 2 common neighbours.

An example with $m = 3$ and $n = 3$

Let $G = \langle x \rangle$, where $x^3 = 1$.

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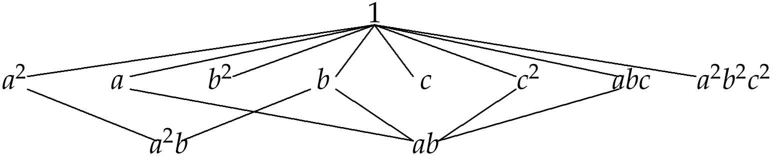

Vertices 1 and ab are at distance 2, and have 4 common neighbours. Vertices 1 and a^2b are at distance 2, and have 2 common neighbours. So the graph is not distance-regular.

Eigenvalues of the adjacency matrix

For $i = 0, 1, \ldots, m$, let A_i be the $n \times n$ matrix whose rows and columns are indexed by elements of G with

$$A_i(\alpha, \beta) = \begin{cases} 1 & \text{if } \alpha \text{ and } \beta \text{ are in the same part of } Q_i \text{ but } \alpha \neq \beta, \\ 0 & \text{otherwise.} \end{cases}$$

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Hence each W-subspace is contained in an eigenspace of A.

Eigenvalues of the adjacency matrix, continued

If Q is a partition in the diagonal semi-lattice,
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If $\rho(Q) = k$ and $\mathbf{v} \in W_Q$ then $\mathbf{v}$ is constant on precisely k of the minimal partitions $Q_0, Q_1, \ldots, Q_m$. Hence the eigenvalue of A on $\mathbf{v}$ is

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$$\dim(W_Q) = \sum_{P} \mu(Q,P) n^{m-\rho(P)}.$$

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We managed to prove the following for the diagonal semi-lattice.

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$$\mu(Q,P) = \begin{cases} (-1)^{\rho(P)-\rho(Q)} & \textit{if } Q \preccurlyeq P \textit{ and } P \neq U, \\ (-1)^{m-\rho(Q)}(m-\rho(Q)) & \textit{if } P = U, \\ 0 & \textit{if } Q \not\preccurlyeq P. \end{cases}$$

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This just leaves the subspace W_U of constant vectors, which has eigenvalue $(m+1)(n-1)$ with multiplicity 1.

Chapter 5

… and beyond

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We found an interesting example with $k = 2$ and $|\Omega| = 8^2$ where the four Latin squares are all Cayley tables of groups, but those groups come from three different isomorphism classes.

Mutually orthogonal diagonal semilattices

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It seems obvious that the isomorphism type of $G_{\mathcal{S}}$ should not depend on $\mathcal{S}$, but we have not been able to prove this yet.

Regular mutually orthogonal diagonal semilattices

Let us call a set of MODS regular if the isomorphism type of $G_{\mathcal{S}}$ does not depend on $\mathcal{S}$.

Theorem

If $m \geq 3$ and $k \geq 2$ then the unique (up to isomorphism) group G defined by a regular set of MODS is Abelian. Furthermore, G admits three fixed-point-free automorphisms whose product is the identity.

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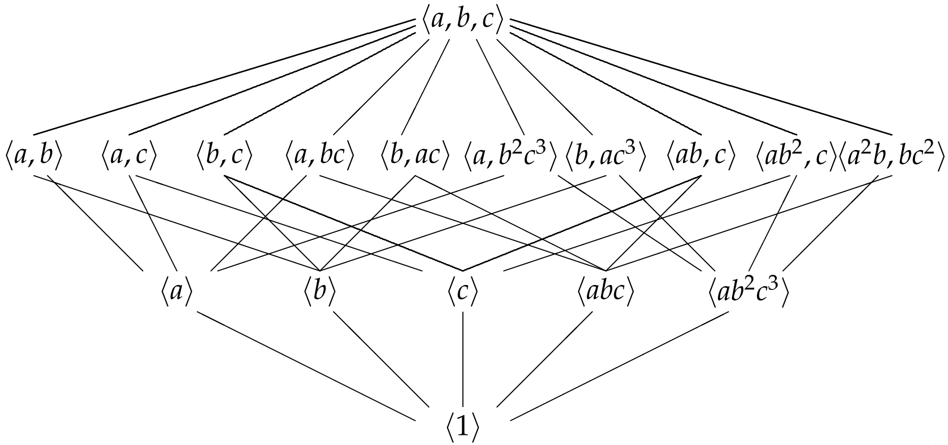
One way of construcing orthogonal arrays uses elementary Abelian groups, building on the methods used in fractional factorial designs. Taking the dual of such a group (in the algebraic sense) gives the dual concept in the partition sense, which is what we want.

Some subgroups of an elementary Abelian group

If p is prime and $p \geq 5$ we can make a MODS with $n = p^3$, $m = 3$ and $k = 2$ by using some subgroups of an elementary Abelian group of order p^3.



Another MODS

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If $G = \langle a, b, c, d \rangle$ then the six subgroups

$$\langle a \rangle,\ \langle b \rangle,\ \langle c \rangle,\ \langle d \rangle,\ \langle abcd \rangle,\ \langle ab^2c^3d^4 \rangle$$

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Unfortunately, my slide is too narrow to contain the Hasse diagram.

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Theorem

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When $m = 2$, this theorem specializes to the well-known upper bound for the number of mutually orthogonal Latin squares of order n.

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