Designs for variety trials with very low replication

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Abstract

In the early stages of testing new varieties, it is common that there are only small quantities of seed of many new varieties.

In the UK (and some other countries with centuries of agriculture on the same land) variation within a field can be well represented by a division into blocks.

Even when that is not the case, subsequent phases
(such as testing for milling quality,
or evaluation in a laboratory)
have natural blocks, such as days or runs of a machine.

I will discuss how to arrange the varieties in a block design when the average replication is less than two.

Variety Testing

In breeding trials of new varieties, typically there is very little seed of each of the new varieties.

Traditionally, an experiment has one plot for each new variety and several plots for a well-established "control":

	New	Control	Total
for example,	30	1	31
	1	8	38

In the last 12 years, Cullis and colleagues in Australia (and independently Bueno and Gilmour) have suggested replacing many occurrences of the the control by double replicates of a small number of new varieties:

	New	New	Control	Total
for example,	24	6	1	31
	1	2	2	38

This is an improvement if there are no blocks.

How do we allow for variation between the plots?

"…on any given field agricultural operations, at least for centuries, have followed one of two directions, which are usually those of the rows and columns; consequently streaks of fertility, weed infestation, etc., do, in fact, occur predominantly in those two directions."

R. A. Fisher,
letter to H. Jeffreys,
30 May 1938
(selected correspondence edited by J. H. Bennett)

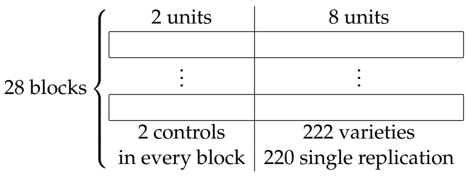
(This assumption is dubious for field trials in Australia.)

If field operations have been primarily in one direction for a long time, then it is reasonable to divide the fields into blocks whose length runs along that direction.

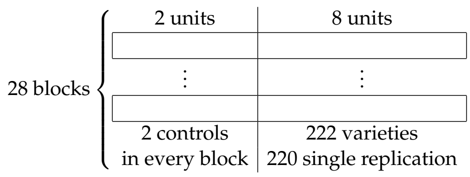
Blocking in the second phase: an example

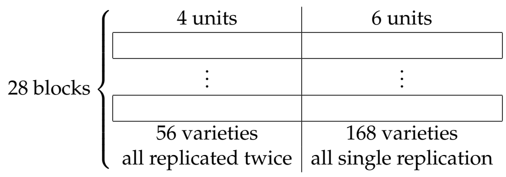
The milling phase of a wheat variety trial has 224 varieties to be compared. Only 10 can be milled in any one day. The trial can take place over 28 days, so there are 28 blocks of size 10.

There are only $280 - 224 = 56$ experimental units "spare" for replication. How should these be allocated?



Two possible designs for 224 varieties in 28 blocks of 10





The problem

We are given b blocks of size k. We are given v varieties.
Assume that

$$\text{average replication} = \bar{r} = \frac{bk}{v} \leq 2.$$

How should we allocate varieties to blocks?

What makes a block design good?

Linear model, estimation and variance

We measure the response Y on each unit in each block.

If that unit has variety i and block D, then we assume that

$$Y = \tau_i + \beta_D + \text{random noise},$$

where the random noise is independently normally distributed with zero mean and constant variance σ^2.

We want to estimate all the simple differences $\tau_i - \tau_j$.

Put

$$V_{ij}\sigma^2 = \text{variance of the best linear unbiased estimator for } \tau_i - \tau_j.$$

We want all the V_{ij} to be small.

Optimality

Apart from the constant multiple σ^2,

$$V_{ij} = \text{variance of the BLUE for } \tau_i - \tau_j.$$

Put

$$V_T = \sum_{i=1}^{v-1} \sum_{j=i+1}^{v} V_{ij} = \text{sum of variances of variety differences.}$$

Definition
For given values of b (the number of blocks),
k (the size of the blocks) and v (the number of varieties),
a block design is A-optimal if it minimizes V_T.

An example with $5n + 10$ varieties in 5 blocks of size $4 + n$

1	2	3	4	A_1	$\cdots$	A_n
1	5	6	7	B_1	$\cdots$	B_n
2	5	8	9	C_1	$\cdots$	C_n
3	6	8	0	D_1	$\cdots$	D_n
4	7	9	0	E_1	$\cdots$	E_n

How do we calculate pairwise variances in a generic design?

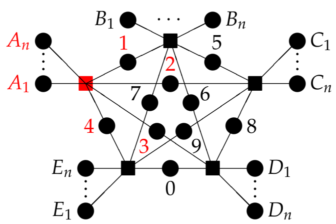
Levi graph

The Levi graph of the block design has

- one vertex for each variety
- one vertex for each block
- one edge for each plot (a.k.a. experimental unit), so that the edge for plot ω joins the vertex for the variety on ω to the vertex for the block containing ω.

Levi graph: example

1	2	3	4	A_1	$\cdots$	A_n
1	5	6	7	B_1	$\cdots$	B_n
2	5	8	9	C_1	$\cdots$	C_n
3	6	8	0	D_1	$\cdots$	D_n
4	7	9	0	E_1	$\cdots$	E_n



Electrical networks

We can consider the Levi graph as an electrical network with a 1-ohm resistance in each edge.
Connect a 1-volt battery between vertices i and j.
Current flows in the network, according to these rules.

1. Ohm's Law:
   In every edge,
   voltage drop = current $\times$ resistance = current.
2. Kirchhoff's Voltage Law:
   The total voltage drop from one vertex to any other vertex is the same whichever path we take from one to the other.
3. Kirchhoff's Current Law:
   At every vertex which is not connected to the battery, the total current coming in is equal to the total current going out.

Find the total current I from i to j, then use Ohm's Law to define the effective resistance R_{ij} between i and j as $1/I$.

Electrical networks: variance

Reminder: $V_{ij} =$ variance of BLUE of $\tau_i - \tau_j$ for varieties i and j.

Theorem
If R_{ij} is the effective resistance between variety vertices i and j in the Levi graph then

$$R_{ij} = V_{ij}.$$

Put: $V_{CD} =$ variance of BLUE of $\beta_C - \beta_D$ for blocks C and D,
$V_{iC} =$ variance of BLUE of $\tau_i + \beta_C$ for variety i and block C.

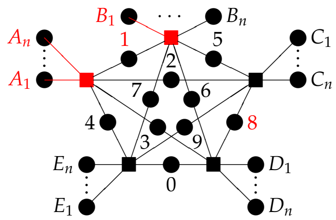
Theorem
If R_{CD} and R_{iC} are the effective resistances between vertices C and D, and between i and C respectively, in the Levi graph then

$$R_{CD} = V_{CD} \quad \text{and} \quad R_{iC} = V_{iC}.$$

Effective resistances are easy to calculate without matrix inversion if the graph is sparse.

Pairwise resistance

(remove $A_1, \ldots, E_n$ to get core subdesign Γ)



$$\begin{aligned}
\text{Resistance}(A_1, A_2) &= 2 \\
\text{Resistance}(A_1, B_1) &= 2 + \text{Resistance}(\text{block } A, \text{block } B) \text{ in } \Gamma \\
\text{Resistance}(A_1, 8) &= 1 + \text{Resistance}(\text{block } A, 8) \text{ in } \Gamma \\
\text{Resistance}(1, 8) &= \text{Resistance}(1, 8) \text{ in } \Gamma
\end{aligned}$$

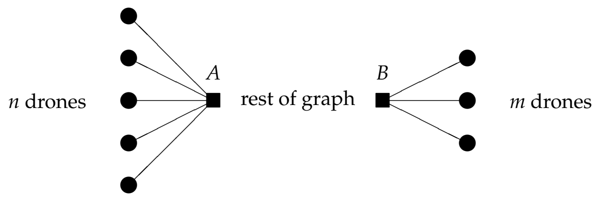
Silly names just for this talk

Definition
Call a variety a

- a drone if it has replication 1;
- a queen-bee if it occurs in every block;
- a worker otherwise.

Is it better to put all the drones into one block (or a few blocks), or are they better distributed equally among all the blocks?

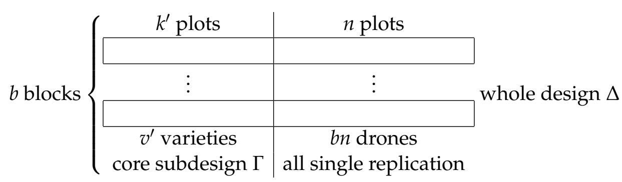
How should we distribute the drones?



If we move all the drones in block B into block A then we reduce nm variances from $2 + R_{AB}$ to 2.

Then we have to remove m non-drones from block A, and this increases the resistance between A and the rest of the graph. This increases the variances between these $n+m$ drones and the remaining $v - n - m$ varieties. This more than compensates for the original reduction in variance.

From now on, distribute drones as equally as possible



Whole design Δ has v treatments in b blocks of size $k = k' + n$;
the core subdesign Γ has v' core varieties in b blocks of size k',
where $v' = v - bn$.
(The core varieties may include up to $b - 1$ extra drones.)

Sum of the pairwise variances

Theorem (cf. Herzberg and Jarrett, 2007)

If there are n drones in each block of Δ, and the core subdesign Γ has v' varieties in b blocks of size k' then the sum of the variances of variety differences in Δ

$$= V_T(\Delta) = bn(bn + v' - 1) + V_T + nV_{BT} + n^2 V_B,$$

where

$$\begin{aligned} V_T &= \text{the sum of the variances of variety differences in } \Gamma \\ V_B &= \text{the sum of the variances of block differences in } \Gamma \\ V_{BT} &= \text{the sum of the variances of sums of one treatment and one block in } \Gamma. \end{aligned}$$

Sum of variances in whole design if Γ is equi-replicate

$$V_T(\Delta) = bn(bn + v' - 1) + V_T + nV_{BT} + n^2 V_B$$

$$\begin{aligned} V_T &= \text{the sum of the variances of variety differences in } \Gamma \\ V_B &= \text{the sum of the variances of block differences in } \Gamma \\ V_{BT} &= \text{the sum of the variances of sums of one treatment and one block in } \Gamma. \end{aligned}$$

If Γ is equi-replicate with replication r' then

$$\begin{aligned} \frac{k'}{b} V_B - b &= \frac{r'}{v'} V_T - v'; \\ V_{BT} &= \frac{2b}{v'} V_T + \frac{v'}{k'}(b - v' - 1), \end{aligned}$$

and so V_B and V_{BT} are both increasing functions of V_T.

Consequence

For a given choice of k', use the core subdesign Γ which minimizes V_T.

Sum of variances in whole design if there are many drones

$$V_T(\Delta) = bn(bn + v' - 1) + V_T + nV_{BT} + n^2 V_B$$

$$\begin{aligned} V_T &= \text{the sum of the variances of variety differences in } \Gamma \\ V_B &= \text{the sum of the variances of block differences in } \Gamma \\ V_{BT} &= \text{the sum of the variances of sums of one treatment and one block in } \Gamma. \end{aligned}$$

Consequence

If n is large, we need to focus on reducing V_B, so it may be best to increase the number of drones and decrease k' (the size of blocks in the core subdesign Γ), so that average replication within Γ is more than 2.

An example of this non-intuitive result

If there are $4(2+n)$ varieties in 4 blocks of size $4+n$, the design on the left is A-better than the design on the right if and only if $n < 50$.

1	2	3	4	n drones
1	2	5	6	n drones
3	6	7	8	n drones
4	5	7	8	n drones

1	2	3	$n+1$ drones
1	2	4	$n+1$ drones
1	3	4	$n+1$ drones
2	3	4	$n+1$ drones

A definite result

Theorem (Duals of BIBDs cannot be beaten)

Suppose that we are given b blocks of size k, and v varieties. For $i = 1, 2$, let design Δ_i have core subdesign Γ_i with block size k_i. If Γ_1 is the dual of a balanced incomplete block design and $k_1 > k_2$ then Δ_2 is worse than Δ_1 on the A criterion, no matter how big v is.

Design	Δ_1		Δ_2
Core subdesign	Γ_1		Γ_2
block size in subdesign	k_1	$>$	k_2
property of core subdesign	dual of BIBD		arbitrary

then Δ_1 is A-better than Δ_2.

An example of the good result

If there are $7(3+n)$ varieties in 7 blocks of size $6+n$, the design on the left is A-better than the design on the right, for all values of n.

1	2	3	4	5	6	n drones
1	7	8	9	10	11	n drones
2	7	12	13	14	15	n drones
3	8	12	16	17	18	n drones
4	9	13	16	19	20	n drones
5	10	14	17	19	21	n drones
6	11	15	18	20	21	n drones

1	4	6	7	$n+2$ drones
1	2	5	7	$n+2$ drones
1	2	3	6	$n+2$ drones
2	3	4	7	$n+2$ drones
1	3	4	5	$n+2$ drones
2	4	5	6	$n+2$ drones
3	5	6	7	$n+2$ drones

Another example of the good result

If there are $4n+6$ varieties in 4 blocks of size $3+n$,
the design on the left is A-better than the design on the right,
for all values of n.

1	2	3	n drones
1	4	5	n drones
2	4	6	n drones
3	5	6	n drones

1	2	$n+1$ drones
1	2	$n+1$ drones
1	2	$n+1$ drones
1	2	$n+1$ drones

Strategy

Given b, v and k, how do we find an A-optimal design
for v varieties in b blocks of size k when

$$\frac{bk}{2} \leq v \leq b(k-1)+1?$$

Average replication ≤ 2 Maximum v for estimability

Case 1. $b=2$ or $b=3$ (very small b).
Case 2. $v=b(k-1)+1$ or $v=b(k-1)$ (very large v).
Case 3. $k_0 \geq b-1$.
Case 4. $2<k_0<b-1$ (small k_0 but not Case 2).

$$k_0 = k - \left\lfloor \frac{2v-bk}{b} \right\rfloor = \text{biggest space per block for non-drones.}$$

Case 1. Only 2 blocks, of size k

Morgan and Jin (2007) showed that the A-optimal designs are
those with $2n$ drones and q queen bees,
where $n=n_0=v-k$ and $q=k'=k_0=k-n_0=2k-v$.

queens						drones				
1	2	3	4	...	q	A_1	A_2	A_3	...	A_n
1	2	3	4	...	q	B_1	B_2	B_3	...	B_n

Case 1 continued. 3 blocks of size k

Using the result about drone-distribution and
the nice theorem about duals of BIBDs, RAB has shown that
the A-optimal designs are as follows when v is divisible by 3
(and presumably small changes deal with the other cases).
There are $3w$ workers and $3n$ drones,
where $3w=3k-v$ and $n=n_0=k-2w$ and $k'=k_0=2w$.

w copies of design using all pairs from 3							drones				
1	2	4	5	...	$3w-2$	$3w-1$	A_1	A_2	A_3	...	A_n
1	3	4	6	...	$3w-2$	$3w$	B_1	B_2	B_3	...	B_n
2	3	5	6	...	$3w-1$	$3w$	C_1	C_2	C_3	...	C_n

Case 2. $v=b(k-1)+1$

This is the maximum number of varieties that can be tested in
b blocks of size k with all comparisons estimable.

Mandal, Shah and Sinha (1991), for $k=2$,
Dean and co-authors,
and Bailey and Cameron (2013), for general block size,
showed that, no matter how many blocks there are,
the A-optimal design has the following form.

1 queen	$v-1$ drones				
1	A_1	A_2	A_3	...	A_{k-1}
1	B_1	B_2	B_3	...	B_{k-1}
1	C_1	C_2	C_3	...	C_{k-1}
1	D_1	D_2	D_3	...	D_{k-1}
1	E_1	E_2	E_3	...	E_{k-1}

Case 2 continued. $v=b(k-1)$

The A-optimal designs were found for all cases
by Krafft and Schaefer (1997).

small k and b — chain

1	2	A_1
2	3	B_1
3	4	C_1
4	5	D_1
5	6	E_1
6	1	F_1

increase k if $b \geq 5$ — smaller chain

1	2	A_1	A_2
2	3	B_1	B_2
3	1	C_1	C_2
1	D_1	D_2	D_3
1	E_1	E_2	E_3
1	F_1	F_2	F_3

then increase b — 1 queen

1	2	A_1	A_2
1	2	B_1	B_2
1	C_1	C_2	C_3
1	D_1	D_2	D_3
1	E_1	E_2	E_3
1	F_1	F_2	F_3
1	G_1	G_2	G_3

Youden and Connor (1953) had recommended chain designs.

Case 2. $v = b(k-1)$ revisited

Theorem

Consider a design with b blocks of size 2. For $2 \leq s \leq b$, let Γ_s be the design consisting of a chain of length s, one of whose varieties is in all blocks outside the chain, while all other varieties are drones. Then

$$V_B(\Gamma_s) = \frac{1}{6}[-s^3 + 2bs^2 - (6b-4)s + 6b^2 - 5b].$$

Consequence

1. *If $b = 3$ then $V_B(\Gamma_2) > V_B(\Gamma_3)$ so there is no need for queens.*
2. *If $b = 4$ then $V_B(\Gamma_2) = V_B(\Gamma_4) < V_B(\Gamma_3)$, but $V_T(\Gamma_2) > V_T(\Gamma_4)$ and $V_{BT}(\Gamma_2) > V_{BT}(\Gamma_4)$, so do not use Γ_2 or Γ_3 (no need for queens).*
3. *If $b \geq 5$ then $V_B(\Gamma_2) < V_B(\Gamma_3) < \cdots < V_B(\Gamma_b)$, so we need to use smaller chains as v gets larger.*

Case 3. $k \geq k_0 \geq b-1$

For simplicity, assume that b divides $2v$, so that

$$n_0 = \frac{2v - bk}{b} = \text{minimum number of drones per block.}$$

Then
$$\frac{b(2k-b+1)}{2} \geq v \geq \frac{bk}{2} \geq \frac{b(b-1)}{2}.$$

Let Γ_0 be the design for $b(b-1)/2$ varieties
replicated twice in b blocks of size $b-1$
in such a way that
there is one variety in common to each pair of blocks.
This is A-optimal for these numbers.

Case 3 continued. $k_0 \geq b-1$

$n_0 =$ minimal number of drones per block.

Construction Method

1. *put n_0 drones in each block;*
2. *put in one copy of Γ_0;*
3. *put in as many further copies of Γ_0 as possible (if this uses up all the space, then the nice theorem shows that this is A-optimal);*
4. *in any remaining space, use a good design for workers with replication 2 (so long as there is at least one copy of Γ_0, it probably doesn't make much difference which one is used).*

Case 3. Example: $b = 8$ and $k = 15$ (so $60 \leq v \leq 92$)

60 varieties: all workers ($n_0 = 0$)

1	2	3	4	5	6	7	29	30	31	32	33	34	35	57
1	8	9	10	11	12	13	29	36	37	38	39	40	41	57
2	8	14	15	16	17	18	30	36	42	43	44	45	46	58
3	9	14	19	20	21	22	31	37	42	47	48	49	50	58
4	10	15	19	23	24	25	32	38	43	47	51	52	53	59
5	11	16	20	23	26	27	33	39	44	48	51	54	55	59
6	12	17	21	24	26	28	34	40	45	49	52	54	56	60
7	13	18	22	25	27	28	35	41	46	50	53	55	56	60
one copy of Γ_0							another copy of Γ_0							

Case 3. Example: $b = 8$ and $k = 15$ (so $60 \leq v \leq 92$)

76 varieties: 44 workers, 32 drones ($n_0 = 4$)

1	2	3	4	5	6	7	29	30	31	32	A_1	A_2	A_3	A_4
1	8	9	10	11	12	13	33	34	35	36	B_1	B_2	B_3	B_4
2	8	14	15	16	17	18	37	38	39	40	C_1	C_2	C_3	C_4
3	9	14	19	20	21	22	41	42	43	44	D_1	D_2	D_3	D_4
4	10	15	19	23	24	25	29	33	37	41	E_1	E_2	E_3	E_4
5	11	16	20	23	26	27	30	34	38	42	F_1	F_2	F_3	F_4
6	12	17	21	24	26	28	31	35	39	43	G_1	G_2	G_3	G_4
7	13	18	22	25	27	28	32	36	40	44	H_1	H_2	H_3	H_4
Γ_0							16 workers replication 2				drones			

Case 3. Example: $b = 8$ and $k = 15$ (so $60 \leq v \leq 92$)

92 varieties: 28 workers, 64 drones ($n_0 = 8$)

1	2	3	4	5	6	7	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8
1	8	9	10	11	12	13	B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8
2	8	14	15	16	17	18	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
3	9	14	19	20	21	22	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8
4	10	15	19	23	24	25	E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8
5	11	16	20	23	26	27	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8
6	12	17	21	24	26	28	G_1	G_2	G_3	G_4	G_5	G_6	G_7	G_8
7	13	18	22	25	27	28	H_1	H_2	H_3	H_4	H_5	H_6	H_7	H_8
Γ_0							drones							

Case 3. A bad example: $b = 4$ and $k_0 = 4$

$v = 4n + 8$ and $k = 4 + n$

My strategy gives

1	2	3	7	A_1	...	A_n
1	4	5	7	B_1	...	B_n
2	4	6	8	C_1	...	C_n
3	5	6	8	D_1	...	D_n
Γ_0				drones		

which is worse than

1	2	3	A_1	...	A_{n+1}
1	2	4	B_1	...	B_{n+1}
1	3	4	C_1	...	C_{n+1}
2	3	4	D_1	...	D_{n+1}
$k' = 3$ rep 3			drones		

when $n \geq 50$.

Case 4. $2 < k_0 < b - 1$

For various values of $k_i \leq k_0$,
find the best core subdesign Γ_i for v_i' varieties in b blocks of size k_i. (For equi-replicate core subdesigns,
it is often easier to find the best dual design, which is obtained by interchanging the roles of blocks and varieties.)

$V_T(\Gamma_i)$ = the sum of the variances of variety differences in Γ_i

$V_B(\Gamma_i)$ = the sum of the variances of block differences in Γ_i

$V_{BT}(\Gamma_i)$ = the sum of the variances of sums of one treatment and one block in Γ_i.

If there are n_i drones in each block then, in the whole design Δ,

$$V_T(\Delta) = bn_i(bn_i + v_i' - 1) + V_T(\Gamma_i) + n_i V_{BT}(\Gamma_i) + n_i^2 V_B(\Gamma_i).$$

Use this formula to find the core subdesign which gives the smallest $V_T(\Delta)$.
As the number of varieties increases, it becomes more important to choose Γ_i with a small value of $V_B(\Gamma_i)$.

Case 4 continued. $k_0 = 4 < b - 1$, $V_B(\Gamma_i) \div b(b-1)/2$

Best design for b blocks known to RAB

	Γ_1	Γ_2	Γ_3	Γ_4
k_i	2	3	3	4
	2 queens, both boring	2 queens, 2 workers (rep 2)	b workers rep 3	$2b$ workers rep 2
$b = 6$	1	1^-	0.85	0.87
$b = 7$	1	1^-	0.86	0.92
$b = 8$	1	1^-	0.89	0.93
$b = 9$	1	1^-	0.92	
$b = 10$	1	1^-		
$b = 11$	1	1^-		
$b = 12$	1	1^-	0.98	
$b = 13$	1	1^-	1	1.07
$b = 14$	1	1^-		
$b = 15$	1	1^-	1.01	1.08

As v increases, Γ_3 becomes better than Γ_4.
If $b \geq 14$, then, as v increases, Γ_1 and Γ_2 become better than Γ_3.

Case 4 continued. $2 < k_0 < b - 1$ when $b = 8$: $k_0 = 6$

$k = k_0 = 6$, and 24 varieties, all workers, all replicated twice.

1	2	3	4	5	6
7	8	9	10	11	12
1	7	13	14	15	16
2	8	17	18	19	20
3	9	13	17	21	22
4	10	14	18	23	24
5	11	15	19	21	23
6	12	16	20	22	24

(One worker for each pair of blocks
except for $\{A,B\}$, $\{C,D\}$, $\{E,F\}$ and $\{G,H\}$.)

Case 4 continued. $k = 5$ and $k = 6$ when $b = 8$: $k_0 = 5$

$k = 5$
20 varieties:
20 workers, no drones

1	2	3	4	5
6	7	8	9	10
1	11	12	13	14
2	6	15	16	17
3	7	11	18	19
4	8	12	15	20
5	9	13	16	18
10	14	17	19	20

$k = 6$
28 varieties:
20 workers, 8 drones

1	2	3	4	5	A_1
6	7	8	9	10	B_1
1	11	12	13	14	C_1
2	6	15	16	17	D_1
3	7	11	18	19	E_1
4	8	12	15	20	F_1
5	9	13	16	18	G_1
10	14	17	19	20	H_1

Case 4 continued. $k = 5$ and $k = 6$ when $b = 8$: $k_0 = 4$

$k = 5$
24 varieties:
16 workers, 8 drones

1	2	3	4	A_1
5	6	7	8	B_1
9	10	11	12	C_1
13	14	15	16	D_1
1	5	9	13	E_1
2	6	10	14	F_1
3	7	11	15	G_1
4	8	12	16	H_1
$k' = 4$ rep = 2				

$k = 6$
32 varieties:
8 workers, 24 drones

1	2	4	A_1	A_2	A_3
2	3	5	B_1	B_2	B_3
3	4	6	C_1	C_2	C_3
4	5	7	D_1	D_2	D_3
5	6	8	E_1	E_2	E_3
6	7	1	F_1	F_2	F_3
7	8	2	G_1	G_2	G_3
8	1	3	H_1	H_2	H_3
$k' = 3$ rep 3					

Case 4 continued. $k = 5$ and $k = 6$ when $b = 8$: $k_0 = 3$

$k = 5$
28 varieties:
12 workers, 16 drones

1	2	3	A_1	A_2
1	4	5	B_1	B_2
4	6	7	C_1	C_2
6	8	9	D_1	D_2
2	8	10	E_1	E_2
5	10	11	F_1	F_2
7	11	12	G_1	G_2
3	9	12	H_1	H_2

$k = 6$
36 varieties:
12 workers, 24 drones

1	2	3	A_1	A_2	A_3
1	4	5	B_1	B_2	B_3
4	6	7	C_1	C_2	C_3
6	8	9	D_1	D_2	D_3
2	8	10	E_1	E_2	E_3
5	10	11	F_1	F_2	F_3
7	11	12	G_1	G_2	G_3
3	9	12	H_1	H_2	H_3

Health Warning

The overall message is that there can be phase changes as the spare capacity for replication $(bk - v)$ decreases.
Therefore it is necessary to compare core subdesigns Γ_i with different block size k_i.

Although this overall message is correct,
no one has checked the arithmetic in the examples presented,
so individual cases may be wrong.

Warning for specialists in optimality

The results given are for A-optimality.

A design is MV-optimal if it minimizes the value of the maximal V_{ij}.
If $2v - bk \leq k - 1$ then it is possible to put all the drones in a single block, and the MV-optimal design may have this form.

A design is D-optimal if it minimizes the volume of the confidence ellipsoid for the vector of fitted values $(\tau_1, \tau_2, \ldots)$ under the assumption of independent identically distributed normal errors.
The D-optimal designs have core subdesigns with the largest possible value of k' (and so the smallest possible number of drones), no matter how large the number of varieties.

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