Relations between partitions: some problems

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An example of three uniform partitions of the same set

A	D	G	A	B	C	A	B	C	A	B	C
B	E	H	D	E	F	E	F	D	F	D	E
C	F	I	G	H	I	I	G	H	H	I	G

- The underlying set has size 36 (vegetable patches).
- The partition D into districts has 4 parts of size 9.
- The partition G into gardens has 12 parts of size 3.
- The partition L into letters (lettuce varieties) has 9 parts of size 4.

Three binary relations:

- $G \prec D$, G is a refinement of D;
- $L \perp D$, L is strictly orthogonal to D;
- $L \rhd G$, L is balanced with respect to G.

Some definitions for a uniform partition of a finite set

Ω is the underlying set, of size M.
$V_0 =$ subspace of $\mathbb{R}^\Omega$ consisting of constant vectors.

For a given uniform partition F:

- $n_F =$ number of parts of F;
- $k_F =$ size of each part of F;
- $V_F =$ subspace of $\mathbb{R}^\Omega$ consisting of vectors which are constant on each part of F;
- $V_0 \leq V_F$ and $\dim(V_F) = n_F$;
- X_F is the $M \times n_F$ incidence matrix of elements of Ω in parts of F;
- $P_F = \dfrac{1}{k_F} X_F X_F^\top =$ matrix of orthogonal projection onto V_F, which averages each vector over each part of F.

Some definitions for two uniform partitions of the same set

$N_{FG} = X_F^\top X_G$
$\quad =$ incidence matrix between parts of F and parts of G.

Refinement $F \prec G$ means that
each part of F is contained in a single part of G but $n_F > n_G$.

Orthogonality $F \perp G$ means the following equivalent things:

- $N_{FG} = \text{constant} \times J$;
- $P_F P_G = P_G P_F = P_0 =$ projector onto V_0;
- $(V_F \cap V_0^\perp) \perp (V_G \cap V_0^\perp)$;
- $X_F^\top (I - P_0) X_G = 0$;
- $N_{F0} N_{0G} = k_0 N_{FG} = M N_{FG}$.

Balance If no entry in N_{FG} is bigger than 1
then $F \rhd G$ means the following equivalent things:

- $N_{FG} N_{GF}$ is completely symmetric
(a linear combination of I and J) but not scalar;
- $X_F^\top (I - P_G) X_F$ is completely symmetric but not zero.

(We usually exclude orthogonality.)

Ternary relations?

So you can have fun making sets of partitions on the same set such that each pair is related by refinement or orthogonality or balance.

But hang on! What happens to orthogonality or balance when we project orthogonally away from any partition subspace?

Adjusted orthogonality Partitions F and G have adjusted orthogonality with respect to partition L if

$$X_F^\top (I - P_L) X_G = 0;$$

equivalently,

$$N_{FL} N_{LG} = k_L N_{FG}.$$

Or higher order?

If $\mathcal{L}$ is a set of partitions of Ω, put

$$P_{\mathcal{L}} = \text{matrix of orthogonal projection onto } \sum_{L \in \mathcal{L}} V_L.$$

Adjusted orthogonality Partitions F and G have adjusted orthogonality with respect to the set $\mathcal{L}$ of partitions if

$$X_F^\top (I - P_{\mathcal{L}}) X_G = 0.$$

Triple arrays

	0	2	6	7	8	X
1	B	A	E	D	J	F
4	G	H	B	I	D	E
9	J	I	A	B	C	G
5	F	J	H	C	E	I
3	H	D	C	F	G	A

An $r \times c$ **triple array** is an $r \times c$ rectangle, each cell containing one of $r + c - 1$ letters, such that

- rows R are strictly orthogonal to columns C, with all intersections of size 1;
- rows are balanced with respect to letters (L) (every pair of rows has the same number of letters in common);
- columns are balanced with respect to letters;
- rows and columns have adjusted orthogonality with respect to L (the set of letters in each row has constant size of intersection with the set of letters in each column).

How did I make it? Start with a SBIBD

	A	B	C	D	E	F	G	H	I	J
1	2	3	4	5	6	7	8	9	X	0
4	5	6	7	8	9	X	0	1	2	3
9	X	0	1	2	3	4	5	6	7	8
5	6	7	8	9	X	0	1	2	3	4
3	4	5	6	7	8	9	X	0	1	2

	0	2	6	7	8	X
1				BDF		
4						
9						
5						
3						
column name is in	B	A	A	B	C	A
	F	D	B	C	D	E
	G	H	C	D	E	F
	H	I	E	F	G	G
	J	J	H	I	J	I

A	B	D	E	F	J	row name is not in
B	D	E	G	H	I	
A	B	C	G	I	J	
C	E	F	H	I	J	
A	C	D	F	F	H	

Put the letters in cells and obtain these subsets in rows and columns

Problem: can you do it?

Given a subset of letters allowed for each cell, is it possible to choose an **array of distinct representatives**, one per cell, so that no letter is repeated in a row or column?

Fon-der-Flaass, 1997: the general problem is NP-complete.

Suppose the allowable subsets come from an SBIBD in the way I showed?

- Not if the allowable subsets have size ≤ 2.
- Agrawal (1966): "always possible in the examples tried by the author".
- Rhagavarao and Nageswararao (1974): two false proofs.
- Seberry (1979); Street (1981); Bagchi (1996); Preece, Wallis and Yucas (2005) gave explicit constructions for $q \times (q+1)$ when q is an odd prime power and $q > 3$.
- Computer search always gives a positive result quickly.

Your task: Proof or counter-example.

Balance among three or more uniform partitions

$$P_F = \text{matrix of orthogonal projection onto } V_F$$

$$P_0 = \text{matrix of orthogonal projection onto } V_0$$

$$\text{Put } Q_F = P_F - P_0.$$

F is balanced with respect to G means (in addition to banning orthogonality) $N_{FG}N_{GF}$ is completely symmetric but not scalar; equivalently $X_F^\top(I - P_G)X_F$ is completely symmetric but not zero; equivalently, $Q_FQ_GQ_F$ is a non-zero scalar multiple of Q_F.

If $\mathcal{G}$ is a set of partitions of Ω,

$$P_{\mathcal{G}} = \text{matrix of orthogonal projection onto } \sum_{G \in \mathcal{G}} V_G.$$

F is **balanced with respect to $\mathcal{G}$** if $X_F^\top(I - P_{\mathcal{G}})X_F$ is completely symmetric but not zero.

Exactly three partitions

Suppose that partitions F, G and H each have n parts of size k, and that each pair are balanced (both ways).

Then F is balanced with respect to $\{G, H\}$ if and only if

$$N_{FG}N_{GH}N_{HF} + N_{FH}N_{HG}N_{GF} \text{ is completely symmetric.}$$

Equivalently,

$$Q_F(Q_GQ_H + Q_HQ_G)Q_F \text{ is a non-zero multiple of } Q_F.$$

The above is implied by this stronger condition:

$$N_{FG}N_{GH} \text{ is a linear combination of } N_{FH} \text{ and } J.$$

My attempt at a general definition

A set $\mathcal{L}$ of uniform partitions of Ω, all with n parts, has **universal balance** if whenever $F \in \mathcal{L}$ and $\mathcal{G} \subseteq \mathcal{L} \setminus \{F\}$ then F is balanced with respect to $\mathcal{G}$.

Equivalently (I hope), whenever F and $\mathcal{G}$ are as above, then

$$Q_F\left(\sum_{\sigma \in \mathrm{Sym}((r)} Q_{G_{\sigma(1)}}Q_{G_{\sigma(2)}} \cdots Q_{G_{\sigma(r)}}\right)Q_F$$

is a non-zero multiple of Q_F, where $r = |\mathcal{G}|$.

Equivalently, $\sum_\sigma N_{FG_{\sigma(1)}}N_{G_{\sigma(1)}G_{\sigma(2)}} \ldots N_{G_{\sigma(r)}F}$ is ...

Known families, for n parts of size k

$$N_{FG}N_{GH}N_{HF} + N_{FH}N_{HG}N_{GF} \text{ is completely symmetric,}$$

or its generalization.

- $k = n - 1$: remove a common transversal from a set of mutually orthogonal $n \times n$ Latin squares, so that every N is $J - I$.
- $n \equiv 3 \pmod 4$ and $k = (n+1)/2$ or $k = (n-1)/2$: if there is a doubly-regular tournament of size n, its adjacency matrix A satisfies $I + A + A^\top = J$ and $A^2 \in \langle I, A, J \rangle$, then ensure that each N is either $I + A$ or $I + A^\top$ (or A or $A^\top$).
- $n = 2^{2m}$ and $k = 2^{2m-1} + 2^{m-1}$ or $k = 2^{2m-1} - 2^{m-1}$: Cameron has constructions from quadratic forms, and the strong form of the condition is satisfied. (For $n = 16$ and $k = 6$ this involves compatible Clebsch graphs which form an amorphic association scheme.)

Problem: is this all?

Your task

- Find all possible sets of three or more incidence matrices N_{FG} satisfying the conditions.
- For each such set, realise them as incidence matrices of a set of partitions.
- For each such realisation, find another partition with k parts of size n that is orthogonal to all the rest (surprisingly, this often makes the previous part easier).
- What about two such sets, one with n parts of size k, the other with k parts of size n, and every partition in one set orthogonal to every partition in the other set? (If each set has two partitions, this is a double Youden rectangle, so I only require one of the sets to have at least three partitions.)
- Or three or more?