Circular designs with weak neighbour balance

R. A. Bailey

University of St Andrews

QMUL (emerita)

ACCMCC, December 2015

Joint work with Katarzyna Filipiak and Augustyn Markiewicz (Poznan University of Life Sciences), Joachim Kunert (TU Dortmund) and Peter Cameron (St Andrews)

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The design is neighbour-balanced if (i) no treatment follows itself and (ii) every treatment follows every other treatment equally often. Such designs require a large number of plots. Weak neighbour balance, which can often be achieved in fewer plots, replaces (ii) by a combinatorial condition on the incidence matrix for treatments following each other. Familiar combinatorial objects such as doubly regular tournaments, 2-designs, strongly regular graphs and S-digraphs can be used to construct circular designs with weak neighbour balance.

Small example: each treatment comes "once" per block

Wind →

6 ⋮	0	1	2	3	4	5	6
5 ⋮	0	2	4	6	1	3	5
3 ⋮	0	4	1	5	2	6	3
6 ⋮	0	1	2	3	4	5	6
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1 ⋮	0	6	5	4	3	2	1

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RAB and PJC gave some constructions and non-existence results.

A $0, 1$-matrix

If we have a design which is weakly neighbour balanced but not neighbour balanced then S has zero diagonal, some other entries $\lambda - 1$ and some other entries λ. Put

$$A = S - (\lambda - 1)(J - I).$$

Then

- A is not zero;
- all entries of A are in $\{0, 1\}$;
- A has zero diagonal;
- A has constant row-sums and constant column-sums;
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We know something about (some) matrices like this!

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If Type I, then A has $(t - 1)/2$ non-zero entries in each row and column, and so $t \equiv 3 \bmod 4$.
If Type III, then $A^\top A$ is not completely symmetric.

Hooray for Type I

Theorem

If a WNBD is juxtaposed with a NBD and the result is a WNBD, then the starting WNBD either is a NBD or has Type I.

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Number the positions in each block $1, 2, \ldots,$ starting at the windy end.

Theorem

If a WNBD has the property that each numbered position has all treatments equally often, then it either is a NBD or has Type I.

Type I: $A + A^\top$ and $A^\top A$ are both completely symmetric

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		$x \in Z_t$	$x+1$	
$0 \neq y^2 \in Z_t$	$\dots$	xy^2	$(x+1)y^2$	$\dots$

is a WNBD.

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$t = 3$ ✓, but too small to separate direct effects from upwind effects

$t = 7$ ✓, see next slide

Type I and $t = 7$

0	1	2	3	4	5	6
0	2	4	6	1	3	5
0	4	1	5	2	6	3

Type I and $t = 7$

0	1	2	3	4	5	6
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0	1	2	3	4	5	6
0	2	4	6	1	3	5
0	3	6	2	5	1	4
0	4	1	5	2	6	3
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0	6	5	4	3	2	1

$t = 11$ ✓

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$t = 11$ ✓

$t = 15$? RAB tried using A as the incidence matrix of $\mathrm{PG}(3, 2)$ and proved that it is impossible.

Type I and $t = 15$

Reid and Brown give the following doubling construction.

$$A_2 = \begin{pmatrix} A_1^\top & 0_t & A_1 + I_t \\ 1_t^\top & 0 & 0_t^\top \\ A_1 & 1_t & A_1 \end{pmatrix}$$

If A_1 is Type I for t then A_2 is Type I for $2t + 1$.

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Doing this with $t = 7$ gives a doubly regular tournament Γ_2 on 15 vertices with an automorphism π of order 7.
If we can find a Hamiltonian cycle φ which has no edge in common with any of $\pi^i(\varphi)$ for $i = 1, \ldots, 6$, then φ, $\pi(\varphi), \ldots, \pi^6(\varphi)$ make a WNBD.

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RAB tried and failed to do this by hand.

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PJC used GAP, and found 120 solutions.
KF put this A_2 into Mathematica and asked it to find Hamiltonian decompositions.

Question

treatments	t		$2t+1$
		doubling	
matrix	A_1	$\longrightarrow$	A_2
	$\updownarrow$		$\updownarrow$
digraph	Γ_1		Γ_2
	$\uparrow$		$\uparrow$
design	Δ_1		Δ_2

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$$
\begin{array}{cccc}
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& \uparrow & & \uparrow \\
\text{design} & \Delta_1 & & \Delta_2
\end{array}
$$

Could we go directly from Δ_1 to Δ_2?

Type I designs with rows and columns

Suppose that $t \equiv 3 \bmod 4$ and t is a prime power.
Let x be a primitive element of $\mathrm{GF}(t)$.
In the circular sequence

$$(1, x, x^2, x^3, \ldots, x^{t-1})$$

the successive differences give all non-zero elements of $\mathrm{GF}(t)$.

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$$\text{Put } \phi = (x, 1, 0, x^2, x^3, \ldots, x^{t-1}).$$

If $t \neq 3$ then the number of non-zero squares in the successive differences of ϕ is one different from the number of non-squares in the successive differences of ϕ.

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The $t(t-1)/2$ sequences $s\phi + i$
where s is a non-zero square in $\mathrm{GF}(t)$ and $i \in \mathrm{GF}(t)$
give a weakly neighbour-balanced design in which every treatment occurs $(t-1)/2$ times in each numbered position.

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Now we can regard A as the incidence matrix of a 2-design, with blocks labelled so that the diagonal is zero.

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We can also take advantage of symmetry to find a single Hamiltonian cycle whose images under a group of automorphisms of Γ give the blocks of the WNBD.

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If A is itself symmetric then it is the adjacency matrix of a strongly regular graph in which every pair of distinct vertices have the same number of common neighbours (for example, the Shrikandhe graph and the Clebsch graph).

Again, familiar tricks and use of symmetry give us WNBDs.

Type III: $A^\top A - (\lambda - 1)(A + A^\top)$ is completely symmetric, but $A^\top A$ and $(A + A^\top)$ are not

If A_1 has Type I for t treatments then

$$\begin{pmatrix} A_1 & A_1 + I_t & \dots & A_1 + I_t \\ A_1 + I_t & A_1 & \dots & A_1 + I_t \\ \vdots & \vdots & \ddots & \vdots \\ A_1 + I_t & A_1 + I_t & \dots & A_1 \end{pmatrix}$$

has Type III for mt treatments with $\lambda = m(t+1)/4$

and

$$\begin{pmatrix} 0 & 1_t^\top & 0 & 0_t^\top \\ 0_t & A_1 & 1_t & A_1^\top \\ 0 & 0_t^\top & 0 & 1_t^\top \\ 1_t & A_1^\top & 0_t & A_1 \end{pmatrix}$$

has Type III for $2(t+1)$ treatments with $\lambda = (t+1)/2$.

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$t = 3$ leads to the only Type III WNBDs ($t = 6$ and $t = 8$) found by KF and AM.

Type III doubling (or multiplying) constructions

Again, is there a way of going directly from the smaller design to the larger one?