Designs for half-diallel experiments

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Joint work with Peter Cameron (University of St Andrews)
and Dário Ferreira, Sandra S. Ferreira and Célia Nunes
(Universidade de Beira Interior)

A walk around my subject

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I will illustrate each of these conditions when applied to the same two combinatorial objects.

Strongly regular graphs

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(for example, plots in a field).
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Three $\Omega \times \Omega$ real matrices associated with Γ:

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The graph Γ is strongly regular if A^2 is a linear combination of A, I and J but not all pairs are edges.
In this case, the real vector space $\mathbb{R}^\Omega$ is the orthogonal direct sum of subspaces W_0, W_1 and W_2, each of which is (contained in) an eigenspace of A and an eigenspace of J, where W_0 is the one-dimensional subspace spanned by the all-1 vector $\mathbf{u}$.

Design question and statistical issues

We have a set $\mathcal{T}$ of t treatments. We need to choose a design, which is a function $f\colon \Omega \to \mathcal{T}$ allocating treatment $f(\omega)$ to experimental unit ω. How should we choose f?

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When is the choice of best design not affected by the values of γ_0, γ_1 and γ_2?

Two different desirable statistical conditions

Condition 1 We want the variance V_{ij} of the estimator of $\tau_i - \tau_j$ to be the same for all pairs $\{i, j\}$ of distinct treatments.

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Condition 2 We want the linear combination of the Y_ω (for $\omega \in \Omega$) which gives the best estimate of $\tau_i - \tau_j$ (correct on average, smallest variance) to be the same as the best estimator when $\gamma_0 = \gamma_1 = \gamma_2$. This is the difference between the averages for plots with treatment i and those with treatment j.

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Solution The subspace V_T of $\mathbb{R}^\Omega$ consisting of vectors which are constant on each treatment can be orthogonally decomposed as

$$W_0 \oplus (V_T \cap W_1) \oplus (V_T \cap W_2).$$

Combinatorial Structure 1: Partition into Blocks

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If $k = t$ then each block must contain every treatment.
If $k > t$ then something slightly more complicated is needed.

An example of a balanced incomplete-block design

Here is a balanced incomplete-block design with $b = 14$, $k = 4$, $t = 8$ and $\lambda = 3$.

1	3	5	7	2	4	6	8
1	2	5	6	3	4	7	8
1	2	3	4	5	6	7	8
1	4	5	8	2	3	6	7
1	3	6	8	2	4	5	7
1	2	7	8	3	4	5	6
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			2	3	4	2	3	5	2	4	5

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1	3	4
2	3	4

1	3	5
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1	4	5
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I don't want to get bogged down in the statistical details, so I will say no more about this here.

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Since the treatment subspace V_T contains W_0, there are three possibilities.

(a) $V_T \leq W_0 \oplus W_2$.

(b) $V_T \leq W_0 \oplus W_1$.

(c) $V_T \cap W_1$ and $V_T \cap W_2$ are both non-zero, and $V_T = W_0 \oplus (V_T \cap W_1) \oplus (V_T \cap W_2)$.

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More generally, any subset of treatments may be merged into a single treatment. For example,

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Such designs are used when management constraints make it impractical to apply the treatments to the individual plots.

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(c) $V_T \cap W_1$ and $V_T \cap W_2$ are both non-zero, and
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For example, when $b = 4$, $k = 3$, $t = 6$ and $t_1 = 2$ we get

$A1$	$A2$	$A3$	$B1$	$B2$	$B3$	$A1$	$A2$	$A3$	$B1$	$B2$	$B3$

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These are called split-plot designs.

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For example, the aim of the experiment might be to compare different methods for researchers to collaborate when they are unable to meet face-to-face, such as email, online meetings, old-fashioned letters, telephone calls with and without video.

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This is called the triangular graph $T(m)$.
It is strongly regular, and its adjacency matrix A satisfies

$$A^2 = (2m-8)I + (m-6)A + 4J.$$

How to picture the elements of Ω

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6			$\circ$		$\circ$

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$$\circ = \text{vertices joined to vertex } \{3, 5\}$$

Triangular graph: Condition 1

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Then each treatment occurs on $(m-1)/2$ plots, and $\lambda = m - 1$.
In fact, each treatment misses one individual and occurs once with every other individual.

An example with $m = 7$

	1	2	3	4	5	6
2	B					
3	C	D				
4	D	E	F			
5	E	F	G	A		
6	F	G	A	B	C	
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For strongly regular graphs in general, such designs are called balanced colourings of strongly regular graphs.

Triangular Graph: the other Condition

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Condition 2 We want the linear combination of the Y_ω (for $\omega \in \Omega$) which gives the best estimate of $\tau_i - \tau_j$ (correct on average, smallest variance) to be the same as the best estimator when $\gamma_0 = \gamma_1 = \gamma_2$. This is the difference between the averages for plots with treatment i and those with treatment j.

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Since the treatment subspace V_T contains W_0, there are three possibilities.

(a) $V_T \leq W_0 \oplus W_2$.

(b) $V_T \leq W_0 \oplus W_1$.

(c) $V_T \cap W_1$ and $V_T \cap W_2$ are both non-zero, and $V_T = W_0 \oplus (V_T \cap W_1) \oplus (V_T \cap W_2)$.

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In this case, we can do this by using a symmetric Latin square of order m with a single letter on the main diagonal and omitting the main diagonal and plots above the main diagonal.

An example with $m = 8$

	1	2	3	4	5	6	7
2	C						
3	D	E					
4	E	F	G				
5	F	G	A	B			
6	G	A	B	C	D		
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Just as with complete-block designs, any subset of treatments may be merged into a single treatment.

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When $m = 9$ this gives

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3	2	1						
4	3	2	1					
5	4	3	2	1				
6	4	4	3	2	1			
7	3	4	4	3	2	1		
8	2	3	4	4	3	2	1	
9	1	2	3	4	4	3	2	1

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Solution (c) for Condition 2

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Theorem about this solution

Theorem

For $i = 1, \ldots, n$,
let $\mathbf{w}_i$ be the vector whose entries are

$$\begin{cases} 0 & \textit{on all pairs which do not involve an individual of sort } i \\ 1 & \textit{on all pairs which involve a single individual of sort } i \\ 2 & \textit{on all pairs which involve two indiviudals of sort } i \end{cases}$$

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Then

- *The vectors $\mathbf{w}_1, \ldots, \mathbf{w}_n$ span an n-dimensional subspace of $V_T \cap (W_0 \oplus W_1)$.*
- *If $\mathbf{v} \in V_T$ is orthogonal to $\mathbf{w}_i$ for $i = 1, \ldots, n$ then $\mathbf{v} \in W_2$.*

An example with two sorts

Here $m = 9$, $n = 2$, $s_1 = 3$, $s_2 = 6$ and $t = 9$.

	1	2	3	4	5	6	7	8
2	A							
3	A	A						
4	B	C	D					
5	B	C	D	E				
6	D	B	C	F	I			
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$\mathcal{S}_1 = \{1, 2, 3\}$, $\mathcal{T}_1 = \{A\}$ and $t_1 = 1$.

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9	C	D	B	I	G	H	F	E

$\mathcal{S}_1 = \{1, 2, 3\}$, $\mathcal{T}_1 = \{A\}$ and $t_1 = 1$.
$\mathcal{S}_2 = \{4, 5, 6, 7, 8, 9\}$, $\mathcal{T}_2 = \{E, F, G, H, I\}$ and $t_2 = 5$.

An example with two sorts

Here $m = 9$, $n = 2$, $s_1 = 3$, $s_2 = 6$ and $t = 9$.

	1	2	3	4	5	6	7	8
2	A							
3	A	A						
4	B	C	D					
5	B	C	D	E				
6	D	B	C	F	I			
7	D	B	C	G	H	E		
8	C	D	B	H	F	G	I	
9	C	D	B	I	G	H	F	E

$\mathcal{S}_1 = \{1,2,3\}$, $\mathcal{T}_1 = \{A\}$ and $t_1 = 1$.
$\mathcal{S}_2 = \{4,5,6,7,8,9\}$, $\mathcal{T}_2 = \{E,F,G,H,I\}$ and $t_2 = 5$.
$\mathcal{T}_{12} = \{B,C,D\}$ and $t_{12} = 3$.

An example with three sorts

Here $m = 9$, $n = 3$, $s_1 = 1$, $s_2 = 4$, $s_3 = 4$ and $t = 12$.

	1	2	3	4	5	6	7	8
2	A							
3	A	B						
4	A	C	D					
5	A	D	C	B				
6	E	F	G	H	I			
7	E	G	H	I	F	J		
8	E	H	I	F	G	K	L	
9	E	I	F	G	H	L	K	J

An example with three sorts

Here $m = 9$, $n = 3$, $s_1 = 1$, $s_2 = 4$, $s_3 = 4$ and $t = 12$.

	1	2	3	4	5	6	7	8
2	A							
3	A	B						
4	A	C	D					
5	A	D	C	B				
6	E	F	G	H	I			
7	E	G	H	I	F	J		
8	E	H	I	F	G	K	L	
9	E	I	F	G	H	L	K	J

$\mathcal{S}_1 = \{1\}$, $\mathcal{T}_1 = \emptyset$ and $t_1 = 0$.

An example with three sorts

Here $m = 9$, $n = 3$, $s_1 = 1$, $s_2 = 4$, $s_3 = 4$ and $t = 12$.

	1	2	3	4	5	6	7	8
2	A							
3	A	B						
4	A	C	D					
5	A	D	C	B				
6	E	F	G	H	I			
7	E	G	H	I	F	J		
8	E	H	I	F	G	K	L	
9	E	I	F	G	H	L	K	J

$\mathcal{S}_1 = \{1\}$, $\mathcal{T}_1 = \emptyset$ and $t_1 = 0$.
$\mathcal{S}_2 = \{2,3,4,5\}$, $\mathcal{T}_2 = \{B,C,D\}$ and $t_2 = 3$.

An example with three sorts

Here $m = 9$, $n = 3$, $s_1 = 1$, $s_2 = 4$, $s_3 = 4$ and $t = 12$.

	1	2	3	4	5	6	7	8
2	A							
3	A	B						
4	A	C	D					
5	A	D	C	B				
6	E	F	G	H	I			
7	E	G	H	I	F	J		
8	E	H	I	F	G	K	L	
9	E	I	F	G	H	L	K	J

$\mathcal{S}_1 = \{1\}$, $\mathcal{T}_1 = \emptyset$ and $t_1 = 0$.
$\mathcal{S}_2 = \{2, 3, 4, 5\}$, $\mathcal{T}_2 = \{B, C, D\}$ and $t_2 = 3$.
$\mathcal{S}_3 = \{6, 7, 8, 9\}$, $\mathcal{T}_3 = \{J, K, L\}$ and $t_3 = 3$.

An example with three sorts

Here $m = 9$, $n = 3$, $s_1 = 1$, $s_2 = 4$, $s_3 = 4$ and $t = 12$.

	1	2	3	4	5	6	7	8
2	A							
3	A	B						
4	A	C	D					
5	A	D	C	B				
6	E	F	G	H	I			
7	E	G	H	I	F	J		
8	E	H	I	F	G	K	L	
9	E	I	F	G	H	L	K	J

$\mathcal{S}_1 = \{1\}$, $\mathcal{T}_1 = \emptyset$ and $t_1 = 0$.
$\mathcal{S}_2 = \{2,3,4,5\}$, $\mathcal{T}_2 = \{B,C,D\}$ and $t_2 = 3$.
$\mathcal{S}_3 = \{6,7,8,9\}$, $\mathcal{T}_3 = \{J,K,L\}$ and $t_3 = 3$.
$\mathcal{T}_{12} = \{A\}$ and $t_{12} = 1$.

An example with three sorts

Here $m = 9$, $n = 3$, $s_1 = 1$, $s_2 = 4$, $s_3 = 4$ and $t = 12$.

	1	2	3	4	5	6	7	8
2	A							
3	A	B						
4	A	C	D					
5	A	D	C	B				
6	E	F	G	H	I			
7	E	G	H	I	F	J		
8	E	H	I	F	G	K	L	
9	E	I	F	G	H	L	K	J

$\mathcal{S}_1 = \{1\}$, $\mathcal{T}_1 = \emptyset$ and $t_1 = 0$.
$\mathcal{S}_2 = \{2, 3, 4, 5\}$, $\mathcal{T}_2 = \{B, C, D\}$ and $t_2 = 3$.
$\mathcal{S}_3 = \{6, 7, 8, 9\}$, $\mathcal{T}_3 = \{J, K, L\}$ and $t_3 = 3$.
$\mathcal{T}_{12} = \{A\}$ and $t_{12} = 1$. $\mathcal{T}_{13} = \{E\}$ and $t_{13} = 1$.

An example with three sorts

Here $m = 9$, $n = 3$, $s_1 = 1$, $s_2 = 4$, $s_3 = 4$ and $t = 12$.

	1	2	3	4	5	6	7	8
2	A							
3	A	B						
4	A	C	D					
5	A	D	C	B				
6	E	F	G	H	I			
7	E	G	H	I	F	J		
8	E	H	I	F	G	K	L	
9	E	I	F	G	H	L	K	J

$\mathcal{S}_1 = \{1\}$, $\mathcal{T}_1 = \emptyset$ and $t_1 = 0$.
$\mathcal{S}_2 = \{2, 3, 4, 5\}$, $\mathcal{T}_2 = \{B, C, D\}$ and $t_2 = 3$.
$\mathcal{S}_3 = \{6, 7, 8, 9\}$, $\mathcal{T}_3 = \{J, K, L\}$ and $t_3 = 3$.
$\mathcal{T}_{12} = \{A\}$ and $t_{12} = 1$. $\mathcal{T}_{13} = \{E\}$ and $t_{13} = 1$.
$\mathcal{T}_{23} = \{F, G, H, I\}$ and $t_{23} = 4$.

Terminology

For a wide range of structures on the set Ω,
some statisticians call Condition 2 equivalent estimation.

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For a wide range of structures on the set Ω,
some statisticians call Condition 2 equivalent estimation.

Some other statisticians call Condition 2
commutative orthogonal block structure.