Blocking in multi-stage experiments

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Abstract

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Both of these are comparatively straightforward compared to the case where there is one (or more) stage(s) in which the allocation of the treatments to experimental units must be that of an incomplete-block design.

In the talk, I will develop some general principles for good design, along with methods for evaluating competing designs.

Example 1 (Treatments orthogonal to blocks in both stages): the problem

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Treatment factor F has 2 levels, which are applied in Stage 1,
and these can be changed within each batch.

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In Stage 2, the units must be processed in 6 lots of size 6.
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How should we design the experiment?

Example 1: design 1a

Design 1a Align batches with lots,
and allocate the 6 combinations of levels of F and G
in a randomized complete-block design.

	$G1$	$G2$	$G3$
$F1$	a	b	c
$F2$	d	e	f

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c	b	f	c	d	e
d	f	d	e	a	c
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b	e	b	a	f	f
f	c	c	d	b	a
e	d	a	f	e	b

units		treatments		
source	df	source	df	EMS
Mean	1	Mean	1	$6\sigma_B^2 + 6\sigma_L^2 + \sigma^2 + q_0$
Blocks	5			$6\sigma_B^2 + 6\sigma_L^2 + \sigma^2$
Units[B]	30	F	1	$\sigma^2 + q(F)$
		G	2	$\sigma^2 + q(G)$
		$F\#G$	2	$\sigma^2 + q(FG)$
		residual	25	σ^2

σ^2 = variance of experimental units
σ_B^2 = variance of batches σ_L^2 = variance of lots
$q(F)$ = positive semi-definite quadratic form in parameters for levels of F

Example 1: design 1b

Design 1b Cross batches with lots to form a square array, and allocate the 6 combinations of levels of F and G in a Latin square.

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d	a	c	b	f	e
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b	c	e	f	d	a
e	b	d	a	c	f

units		treatments		
source	df	source	df	EMS
Mean	1	Mean	1	$6\sigma^2_B + 6\sigma^2_L + \sigma^2 + q_0$
Batches	5			$6\sigma^2_B + \sigma^2$
Lots	5			$6\sigma^2_L + \sigma^2$
B#L	25	F	1	$\sigma^2 + q(F)$
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c	e	f	d	a	b
d	a	c	b	f	e
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We have lost 5 residual degrees of freedom, and gained nothing.

An easy lesson

Lesson

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and the blocks from the two stages can be aligned,
this gives more residual degrees of freedom
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Brien, Harch, Correll and Bailey (2011) call this "confounding big with big".

Example 2 (Treatments not orthogonal to blocks): the problem

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In Stage 1, these must be processed in 15 batches of size 4.
There are 10 treatments, which are applied in Stage 1,
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Example 2 (Treatments not orthogonal to blocks): the problem

There are 60 experimental units.

In Stage 1, these must be processed in 15 batches of size 4.
There are 10 treatments, which are applied in Stage 1,
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In Stage 2, the units must be processed in 10 lots of size 6.
No further treatment factor is applied in Stage 2.

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How should we design the experiment?

Example 2: the intuitive part

In Stage 1 we can use a balanced incomplete-block design.
This means that
every pair of treatments occur together in the same number (two) of batches,
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E	H	F	H	J	I	J	I	C	J	C	G	D	G	I
C	A	B	G	E	F	B	C	I	A	G	C	E	B	A
H	F	E	J	I	H	D	H	J	F	J	D	I	D	G

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C	A	B	G	E	F	B	C	I	A	G	C	E	B	A
H	F	E	J	I	H	D	H	J	F	J	D	I	D	G

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batches are columns and each treatment in the group occurs in both rows.

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C	A	B	G	E	F	B	C	I	A	G	C	E	B	A
H	F	E	J	I	H	D	H	J	F	J	D	I	D	G

Use each row as a lot in Stage 2.
The treatment information lost to lots is
the same as the information lost to rectangles,
which is part of the information already lost to batches,
so no further information is lost in Stage 2.

A surprising lesson

Lesson

In a nested row-column design,
if the rows within each rectangle have exactly the same treatments
then the loss of information on treatment differences is the same
as it is in the block design obtained by ignoring rectangles and rows.

This was shown independently by Bagchi, Mukhopadhyay and Sinha (1990), Chang and Notz (1990), and Morgan and Uddin (1993).

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In Example 2, the best design for Stage 2 alone
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Principle

If treatments are applied only in Stage 1,
plan the design by starting with the stage in which the block size is smaller.

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1	3	2	1	1	3
2	2	2	3	2	2
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3	1	3	1	3	3
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units		treatments		
source	df	source	df	EMS
Mean	1	Mean	1	$6\sigma_B^2 + 6\sigma_L^2 + \sigma^2 + q_0$
Blocks	5	F	1	$6\sigma_B^2 + 6\sigma_L^2 + \sigma^2 + q(F)$
		residual	4	$6\sigma_B^2 + 6\sigma_L^2 + \sigma^2$
Units[B]	30	G	2	$\sigma^2 + q(G)$
		$F\#G$	2	$\sigma^2 + q(FG)$
		residual	26	σ^2

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The variance for the contrast between levels of F involves
σ_L^2 as well as σ_B^s, so it is larger than it needs to be.

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The variance for the contrast between levels of F involves
σ_L^2 as well as σ_B^2, so it is larger than it needs to be.
There are 4 residual degrees of freedom for testing
the main effect of F, and this cannot be increased.

Example 3: design 3b

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1	3	2	3	2	1
3	1	1	2	2	3
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Lots	5			$6\sigma_L^2 + \sigma^2$
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Main effect of F has smaller variance than before, and same df.

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Main effect of F has smaller variance than before, and same df.
Other treatment effects have same (small) variance,

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2	2	1	1	3	3

units		treatments		
source	df	source	df	EMS
Mean	1	Mean	1	$6\sigma_B^2 + 6\sigma_L^2 + \sigma^2 + q_0$
Batches	5	F	1	$6\sigma_B^2 + \sigma^2 + q(F)$
		residual	4	$6\sigma_B^2 + \sigma^2$
Lots	5			$6\sigma_L^2 + \sigma^2$
B#L	25	G	2	$\sigma^2 + q(G)$
		F#G	2	$\sigma^2 + q(FG)$
		residual	21	σ^2

Main effect of F has smaller variance than before, and same df.
Other treatment effects have same (small) variance, and df reduced from 26 to 21.

Some principles

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If a treatment factor has to be applied to large units such as blocks in one stage, then try to make it orthogonal to blocks in other stages.

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If a treatment factor has to be applied to large units such as blocks in one stage, then it will have relatively few residual degrees of freedom.
In order not to reduce these further,
try to confound the whole of this block term with the same term in other stages.

Example 3: design 3c

Design 3c Make 3 squares of size $(2 \times 2)/3$ by
crossing pairs of batches (shown as rows) with pairs of lots (shown as columns).

$F1$	$G1, G2, G3$	$G1, G2, G3$
$F2$	$G1, G2, G3$	$G1, G2, G3$

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Randomize levels of F to rows within each square;
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Randomize levels of F to rows within each square;
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Remark Squares is the supremum of Batches and Lots:
Squares $=$ Batches $\vee$ Lots.

Example 3: skeleton anova for design 3c

units		treatments		
source	df	source	df	EMS
Mean	1	Mean	1	$6\sigma_B^2 + 6\sigma_L^2 + \sigma_1^2 + q_0$
Squares	2			$6\sigma_B^2 + 6\sigma_L^2 + \sigma_1^2$
Batches[S]	3	F	1	$6\sigma_B^2 + \sigma_1^2 + q(F)$
		residual	2	$6\sigma_B^2 + \sigma_1^2$
Lots[S]	3			$6\sigma_L^2 + \sigma_1^2$
B#L[S]	3			σ_1^2
Units[B,L,S]	24	G	2	$\sigma^2 + q(G)$
		$F\#G$	2	$\sigma^2 + q(FG)$
		residual	20	σ^2

The randomization argument suggests that $\sigma_1^2 \neq \sigma^2$.

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The randomization argument suggests that $\sigma_1^2 \neq \sigma^2$.
Both residual df have decreased, and nothing has been gained.

Example 4 (Treatment factors confounded with blocks in different stages): the problem

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In Stage 1, these must be processed in 6 batches of size 6.
Treatment factor F has 2 levels, which are applied in Stage 1,
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How should we design the experiment?

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Design 4 Cross batches with lots to form a square array;
randomize the 2 levels of F to whole batches,
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This is a standard design, called a strip-plot design or criss-cross design.
Any futher blocking of either batches or lots reduces the already-small residual df for main effects.

More general numbers

Suppose that there are N experimental units.
Stage 1 groups them into b batches of size s;
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we need m rectangles of b/m batches crossed with c/m lots,
with each intersection containing Nm/bc experimental units,
where m divides b, m divides c and bc divides Nm,
so m divides $\gcd\{b, c\}$, and $\gcd\{b, c\}$ divides $Nm/\mathrm{lcm}\{b, c\}$.

Example 5: calculating possible rectangle sizes

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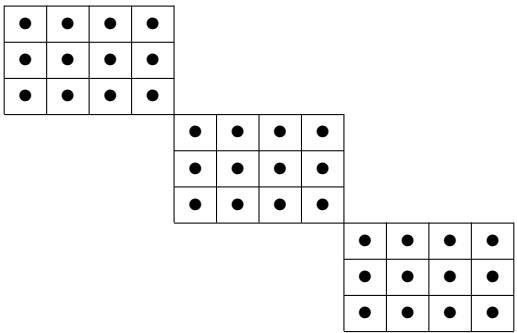
For orthogonality, the only possibililty is three 3×4 rectangles.



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It is desirable to keep m small.

Powers of the same small prime

Suppose that the number of batches and the number of lots are both powers of p, where $p = 2$ or $p = 3$,
that several p-level treatment factors $F_1, F_2, \ldots$ must be applied to whole batches in Stage 1, and several p-level treatment factors $G_1, G_2, \ldots$ must be applied to whole lots in Stage 2.

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For example, if $p = 2$, there are 64 experimental units,
and F_1, F_2, F_3 and F_4 are applied to 16 whole batches in Stage 1
while G_1, G_2 and G_3 are applied to 8 whole lots in Stage 2,
we can use two 4×8 rectangles,
and confound $F_1F_2F_3F_4$ and $G_1G_2G_3$ with each other and with rectangles.

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This technique is called post-fractionation by Bisgaard (1997)
and Vivacqua and Bisgaard (2009).

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- if a treatment factor is confounded with blocks in Stage i, then try to make it orthogonal to blocks in other stages, and try to confound the whole of the Stage i blocks term with a single term in each other stage;

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- if a treatment factor is confounded with blocks in Stage i, then try to make it orthogonal to blocks in other stages, and try to confound the whole of the Stage i blocks term with a single term in each other stage;
- if a treatment factor is neither orthogonal to blocks nor confounded with them, in both Stages 1 and 2, then a nested row-column design may have surprisingly good properties.