A substitute for the non-existent affine plane of order 6

R. A. Bailey
University of St Andrews
QMUL (emerita)

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Joint work with Peter Cameron (University of St Andrews)
and Tomas Nilson (Mid-Sweden University)

Abstract

A Latin square of order n can be used to make an incomplete-block design for n^2 treatments in $3n$ blocks of size n.
The cells are the treatments,
and each row, column and letter defines a block.
Any pair of treatments concur in 0 or 1 blocks, and it is known that the block design is optimal for these parameters.

If there are mutually orthogonal Latin squares, then the process can be continued, eventually giving an affine plane.
But there are no mutually orthogonal Latin squares of order 6, so what should we do if we need a design for 36 treatments in 30 blocks of size 6?

I will describe how a series of mistakes and wrong turnings in a different research project led to an answer.

Outline

1. Square lattice designs.
2. Triple arrays and sesqui-arrays.
3. How the new designs were discovered, part I.
4. Resolvable designs for 36 treatments in blocks of size 6.
5. How the new designs were discovered, part II.
6. As of yesterday, connection with semi-Latin squares.

Chapter 1

Square lattice designs.

Square lattice designs for 16 treatments in 2–4 replicates

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

A	B	C	D
B	A	D	C
C	D	A	B
D	C	B	A

α	β	γ	δ
γ	δ	α	β
δ	γ	β	α
β	α	δ	γ

Replicate 1

1	5	9	13
2	6	10	14
3	7	11	15
4	8	12	16

Replicate 2

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

Replicate 3

1	2	3	4
6	5	8	7
11	12	9	10
16	15	14	13

Replicate 4

1	2	3	4
7	8	5	6
12	11	10	9
14	13	16	15

Using a third Latin square orthogonal to the previous two Latin squares gives a fifth replicate, if required.

All pairwise treatment concurrences are in $\{0, 1\}$.

Square lattice designs for n^2 treatments in rn blocks of n

Square lattice designs were introduced by Yates (1936).
They have n^2 treatments, arranged in r replicates, each replicate consisting of n blocks of size n.

Construction

1. Write the treatments in an $n \times n$ square array.
2. The blocks of Replicate 1 are given by the rows;
   the blocks of Replicate 2 are given by the columns.
3. If $r = 2$ then STOP.
4. Otherwise, write down $r - 2$ mutually orthogonal Latin squares of order n.
5. For $i = 3$ to r, the blocks of Replicate i correspond to the letters in Latin square $i - 2$.

Cheng and Bailey (1991) showed that these designs are optimal among block designs of this size, even over non-resolvable designs.

Side remark

Square lattice designs were independently invented and called nets by Baer in 1939.

Most of the literature on square lattice designs is by people who have never heard of nets, and vice versa.

Efficiency factors and optimality

Given an incomplete-block design for a set $\mathcal{T}$ of treatments in which all blocks have size k and all treatments occur r times, the $\mathcal{T} \times \mathcal{T}$ concurrence matrix Λ has (i,j)-entry equal to the number of blocks in which treatments i and j both occur, and the information matrix is $I - (rk)^{-1}\Lambda$.

The constant vectors are in the kernel of the information matrix.

The eigenvalues for the other eigenvectors are called canonical efficiency factors: the larger the better.

Let μ_A be the harmonic mean of the canonical efficiency factors.

The average variance of the estimate of a difference between two treatments in this design is

$$\frac{1}{\mu_A} \times \begin{array}{l}\text{the average variance in an experiment}\\ \text{with the same resources but no blocks}\end{array}$$

So $\mu_A \leq 1$, and a design maximizing μ_A, for given values of r and k and number of treatments, is A-optimal.

We have a problem when $n = 6$

If $n \in \{2,3,4,5,7,8,9\}$ then there is a complete set of $n-1$ mutually orthogonal Latin squares of order n.

Using these gives a square lattice design for n^2 treatments in $n(n+1)$ blocks of size n, which is a balanced incomplete-block design.

There is not even a pair of mutually orthogonal Latin squares of order 6, so square lattice designs for 36 treatments are available for 2 or 3 replicates only.

Patterson and Williams (1976) used computer search to find a design for 36 treatments in 4 replicates of blocks of size 6.
All pairwise treatment concurrences are in $\{0,1,2\}$.
The value of its A-criterion μ_A is 0.836, which compares well with the unachievable upper bound of 0.840.

Chapter 2

Triple arrays and sesqui-arrays.

Triple arrays

Triple arrays were introduced independently by Preece (1966) and Agrawal (1966), and later named by McSorley, Phillips, Wallis and Yucas (2005).
They are row–column designs with r rows, c columns and v letters, satisfying the following conditions.

- (A1) There is exactly one letter in each row–column intersection.
- (A2) No letter occurs more than once in any row or column.
- (A3) Each letter occurs k times, where $k > 1$ and $vk = rc$.
- (A4) The number of letters common to any row and column is k.
- (A5) The number of letters common to any two rows is the non-zero constant $c(k-1)/(r-1)$.
- (A6) The number of letters common to any two columns is the non-zero constant $r(k-1)/(c-1)$.

A triple array with $r = 4$, $c = 9$, $v = 12$ and $k = 3$

- (A4) The number of letters common to any row and column is $k = 3$.
- (A5) The number of letters common to any two rows is the non-zero constant $c(k-1)/(r-1) = 6$.
- (A6) The number of letters common to any two columns is the non-zero constant $r(k-1)/(c-1) = 1$.

Sterling and Wormald (1976) gave this triple array.

D	H	F	L	E	K	I	G	J
A	K	I	B	J	G	C	L	H
J	A	L	D	B	F	K	E	C
G	E	A	H	I	B	D	C	F

Why triple arrays?

- (A4) The number of letters common to any row and column is $k = 3$.
- (A5) The number of letters common to any two rows is the non-zero constant $c(k-1)/(r-1) = 6$.
- (A6) The number of letters common to any two columns is the non-zero constant $r(k-1)/(c-1) = 1$.
- (A5) Rows are balanced with respect to letters.
- (A6) Columns are balanced with respect to letters.
- (A4) Rows and columns are orthogonal to each other after they have been adjusted for letters.

If letters are blocks, rows are levels of treatment factor $T1$, columns are levels of treatment factor $T2$, and there is no interaction between $T1$ and $T2$, then this is a good design.

Sesqui-arrays are a weakening of triple arrays

Cameron and Nilson introduced the weaker concept of sesqui-array by dropping the condition on pairs of columns. They are row–column designs with r rows, c columns and v letters, satisfying the following conditions.

- (A1) There is exactly one letter in each row–column intersection.
- (A2) No letter occurs more than once in any row or column.
- (A3) Each letter occurs k times, where $k > 1$ and $vk = rc$.
- (A4) The number of letters common to any row and column is k.
- (A5) The number of letters common to any two rows is the non-zero constant $c(k-1)/(r-1)$.

Chapter 3

How the new designs were discovered, part I.

The story: Part I

Consider designs with $n+1$ rows, n^2 columns and $n(n+1)$ letters. Triple arrays have been constructed for $n \in \{3,4,5\}$ by Agrawal (1966) and Sterling and Wormald (1976); for $n \in \{7,8,11,13\}$ by McSorley, Phillips, Wallis and Yucas (2005). There are values of n, such as $n = 6$, for which a BIBD for n^2 treatments in $n(n+1)$ blocks of size n does not exist.

By weakening triple array to sesqui-array, TN and PJC hoped to give a construction for all n.

TN found a general construction, using a pair of mutually orthogonal Latin squares of order n. So this works for all positive integers n except for $n \in \{1,2,6\}$.

This motivated PJC to find a sesqui-array for $n = 6$.

Later, RAB found a simpler version of TN's construction, that needs a Latin square of order n but not orthogonal Latin squares. So $n = 6$ is covered. If this had been known earlier, PJC would not have found the nice design for $n = 6$.

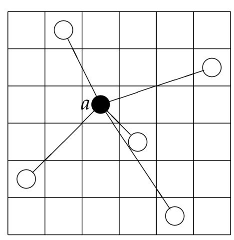
Chapter 4

Resolvable designs for 36 treatments in blocks of size 6.

The Sylvester graph and its starfish

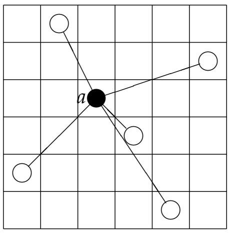
The Sylvester graph Σ is a graph on 36 vertices with valency 5. It has a transitive group of automorphisms, so it looks the same from each vertex.

The vertices can be thought of as the cells of a 6×6 grid.



At each vertex a, the *starfish* $S(a)$ defined by the 5 edges at a has 6 vertices, one in each row and one in each column.

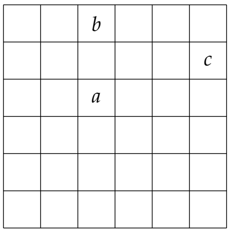
Pedantic naming



When I started to explain these ideas,
I called this set of six vertices the spider centred at a.
PJC pointed out that spiders usually have more than five legs,
whereas some starfish have five.

A real starfish

Starfish whose centres are in the same column



If there is an edge from a to c and an edge from b to c
then the starfish $S(c)$ has two vertices in the third column.
This cannot happen,
so the starfish $S(a)$ and $S(b)$ have no vertices in common.

So, for any one column,
the 6 starfish centred on vertices in that column do not overlap,
and so they give a single replicate of 6 blocks of size 6.

The galaxy of starfish centered on column 3

D	A	B^*	C	E	F
F	E	C^*	B	D	A
E	B	A^*	D	F	C
B	F	D^*	A	C	E
A	C	E^*	F	B	D
C	D	F^*	E	A	B

This is a Latin square.

Constructing resolved designs with r replicates

For $r = 2$ or $r = 3$:

- Replicate 1 the blocks are the rows of the grid
- Replicate 2 the blocks are the columns of the grid
- Replicate 3 the blocks are the starfish of one particular column

These are square lattice designs.

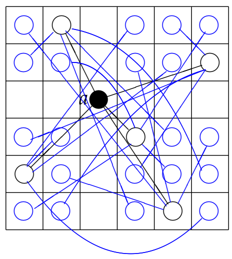
For $r = 4$ or $r = 5$ we can construct very efficient resolved designs using some of

- all rows of the grid
- all columns of the grid
- all starfish of some columns.

Note that, if there is an edge from a to c, then treatments a and c both occur in both starfish $S(a)$ and $S(c)$.
So if we use the galaxies of starfish of two or more columns then some treatment concurrences will be bigger than 1.

The fine details of which designs we chose will be shown later.

More properties of the Sylvester graph



Vertices at distance 2 from a are all in rows and columns different from a.
The Sylvester graph has no triangles or quadrilaterals.

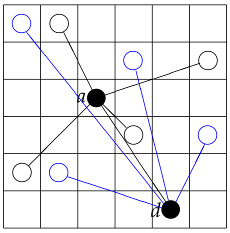
This implies that, if a is any vertex, the vertices at distance 2 from vertex a are precisely those vertices which are not in the starfish $S(a)$ or the row containing a or the column containing a.

Consquence I: concurrences

The Sylvester graph has no triangles or quadrilaterals.

Consequence

If we make each starfish into a block, then the only way that distinct treatments a and d can occur together in more than one block is for vertices a and d to be joined by an edge so that they both occur in the starfish $S(a)$ and $S(d)$.



Consquence II: association scheme

If a is any vertex, the vertices at distance 2 from vertex a are precisely those vertices which are not in the starfish $S(a)$ or the row containing a or the column containing a.

Consequence

The four binary relations:

- different vertices in the same row;
- different vertices in the same column;
- vertices joined by an edge in the Sylvester graph Σ;
- vertices at distance 2 in Σ

form an association scheme.

So, for any incomplete-block design which is partially balanced with respect to this association scheme, the information matrix has five eigenspaces, which we know (in fact, they have dimensions 1, 5, 5, 9 and 16), so it is straightforward to calculate the eigenvalues and hence the canonical efficiency factors.

Constructing a resolved design with 6 replicates

For each column, make a replicate whose blocks are the 6 starfish whose centres are in that column.

$$\text{concurrence} = \begin{cases} 2 & \text{for vertices joined by an edge} \\ 1 & \text{for vertices at distance 2} \\ 0 & \text{for vertices in the same row or column.} \end{cases}$$

canonical efficiency factor	1	$\frac{8}{9}$	$\frac{3}{4}$
multiplicity	10	9	16

The harmonic mean is $\mu_A = 0.8442$.

The unachievable upper bound given by the non-existent square lattice design is $\mu_A = 0.8537$.

Constructing a resolved design with 7 replicates

For each column, make a replicate whose blocks are the 6 starfish whose centres are in that column.
For the 7-th replicate, the blocks are the columns.

$$\text{concurrence} = \begin{cases} 2 & \text{for vertices joined by an edge} \\ 1 & \text{for vertices at distance 2} \\ 1 & \text{for vertices in the same column} \\ 0 & \text{for vertices in the same row.} \end{cases}$$

canonical efficiency factor	1	$\frac{19}{21}$	$\frac{6}{7}$	$\frac{11}{14}$
multiplicity	5	9	5	16

The harmonic mean is $\mu_A = 0.8507$.

The unachievable upper bound given by the non-existent square lattice design is $A = 0.8571$.

Constructing a resolved design with 8 replicates

For each column, make a replicate whose blocks are the 6 starfish whose centres are in that column.
For the 7-th replicate, the blocks are the columns.
For the 8-th replicate, the blocks are the rows.

$$\text{concurrence} = \begin{cases} 2 & \text{for vertices joined by an edge} \\ 1 & \text{otherwise} \end{cases}$$

canonical efficiency factor	$\frac{11}{12}$	$\frac{7}{8}$	$\frac{13}{16}$
multiplicity	9	10	16

The harmonic mean is $\mu_A = 0.8549$.

The non-existent design consisting of a balanced design in 7 replicates with one more replicate adjoined would have $A = 0.8547$.

Compare this with computer search

When Emlyn Williams saw what we had done, he was motivated to re-run that computer search from the 1970s with his current software and hardware.

r	R, C, $*^{r-2}$	C, $*^{r-1}$	$*^r$	HDP/ERW	ERW	square lattice
3	0.8235					0.8235
4	0.8380	0.8341	0.8285	0.836	0.8393	0.8400
5	0.8453	0.8422	0.8383		0.8464	0.8485
6	0.8498	0.8473	0.8442		0.8510	0.8537
7	0.8528	0.8507			0.8542	0.8571
8	0.8549				0.8549	0.8547

Highlighted entries correspond to partially balanced designs.
Our designs have the small advantage that a late decision to add or drop a replicate leaves a design in the same series.

Chapter 5

How the new designs were discovered, part II.

Back to the sesqui-arrays

These wonderful designs are a fortunate byproduct of a wrong turning in the search for sesqui-arrays.

How do we take the one with 7 replicates and turn its dual into a 7×36 sesqui-array with 42 letters?

The story: Part II

RAB: I am typing up some of these new designs. Is your sesqui-array for $n = 6$ written out explicitly?

PJC: Not yet. I will just program GAP to do it for me.

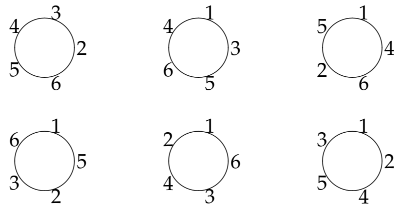
A bit later, PJC: Oh no! My construction does not work after all.
Each column has the correct set of letters,
but their arrangement in rows is wrong,
because each row has some letters occurring 5 times.

	1	2	3	4	5	6	← sets of six columns
$*$	1	2	3	4	5	6	← sets of six letters
1	$*$	1	1	1	1	1	
2	2	$*$	2	2	2	2	
3	3	3	$*$	3	3	3	
4	4	4	4	$*$	4	4	
5	5	5	5	5	$*$	5	
6	6	6	6	6	6	$*$	

Forestry to the rescue

Later, PJC: The only hope of putting this right is to permute the letters in each column. I need 6 permutations. Each fixes the first row and one other. The rest of each permutation gives a circle on the other 5 rows, and I want these circles to have every row following each other row exactly once.

RAB: Easy peasy. That is a neighbour-balanced design for 6 treatments in 6 circular blocks of size 5. I made one of those for experiments in forestry 25 years ago.



Ongoing work

We have indeed constructed that 7×36 sesqui-array, and checked all of its properties very carefully, but it is too large to show on a slide using any font large enough for you to read.

We are still re-checking the calculations to compare different designs for smaller values of r.
This is harder than what I showed, because we cannot use the association scheme if we are not using all starfish.
On the other hand, the calculation is made easier by the fact that, because of the large group of automorphisms,
if we use the starfish from m columns (where $1 \leq m \leq 5$)
it does not matter which subset of m columns we use.
We are also comparing our designs to Emlyn Williams's.
Having the same value of μ_A does not imply isomorphism of the block designs, even when the concurrence graphs are isomorphic.

Chapter 6

Semi-Latin squares.

What is a semi-Latin square?

A $(n \times n)/s$ semi-Latin square is an arrangement of ns treatments in n^2 blocks of size s which are laid out in a $n \times n$ square in such a way that each treatment occurs once in each row and once in each column.

(There is some overlap with Howell designs and with orthogonal multi-arrays. Again, those who use one name are not always aware of literature using one of the other names.)

If a semi-Latin square is made by superposing s mutually orthogonal $n \times n$ Latin squares then it is called a Trojan square and is known to be optimal.

It does not have to be made by superposing Latin squares.

What are the optimal ones when $n = 6$?

From starfish to semi-Latin square

Each galaxy of starfish gives a 6×6 Latin square.
So a block design made with s galaxies can be viewed as a $(6 \times 6)/s$ semi-Latin square.

If the block design has A-criterion μ_A and the semi-Latin square has A-criterion λ_A then

$$\frac{35}{\mu_A} = 6(6-s) + \frac{6s-1}{\lambda_A}.$$

So maximizing μ_A is the same as maximizing λ_A (among semi-Latin squares which are superpositions of Latin squares).

The semi-Latin square made from the galaxies of starfish centered on columns 3 and 4

D	ζ	A	ϵ	B^*	β	C	γ^+	E	δ	F	α
F	δ	E	α	C^*	γ	B	β^+	D	ϵ	A	ζ
E	β	B	ζ	A^*	α	D	δ^+	F	γ	C	ϵ
B	ϵ	F	β	D^*	δ	A	α^+	C	ζ	E	γ
A	γ	C	δ	E^*	ϵ	F	ζ^+	B	α	D	β
C	α	D	γ	F^*	ζ	E	ϵ^+	A	β	B	δ

*centre of Latin starfish $\quad$ $^+$centre of Greek starfish

What is known about good semi-Latin squares with $n = 6$?

Good designs have been found by RAB, Gordon Royle and Leonard Soicher, partly by computer search. Independently, Brickell found some in communications theory. In 2013, LHS gave a $(6 \times 6)/6$ semi-Latin square made superposing Latin squares, so it gives $(6 \times 6)/s$ semi-Latin squares for $2 \le s \le 6$. Its automorphism group has order 144; the automorphism group of the starfish design is $\mathrm{Aut}(S_6)$, of order 1440.

s	$*^s$	Brickell RAB 1990	GR 1997	Brickell LHS web	LHS 2013	Trojan square
2	0.4889	0.5127	0.5133		0.5116	0.5238
3	0.6730			0.6922	0.6745	0.6939
4	0.7604				0.7614	0.7753
5	0.8111				0.8111	0.8227
6	0.8442				0.8442	0.8537

Do the LHS designs for $s = 5$ and $s = 6$ have concurrence graph isomorphic to ours?

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The new stuff

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