Circular designs with weak neighbour balance

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All Kinds of Mathematics Remind of You,
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Joint work with Katarzyna Filipiak and Augustyn Markiewicz (Poznan University of Life Sciences), Joachim Kunert (TU Dortmund) and Peter Cameron (St Andrews)

Abstract

We consider designs where each block is a circle,
or can be considered as such by adjoining border plots.
Sometimes there is the extra complication that
plots are cross-classified in a rectangle where rows are blocks
and the columns are also important.

The design is neighbour-balanced if (i) no treatment follows itself and (ii) every treatment follows every other treatment equally often. Such designs require a large number of plots.

Weak neighbour balance, which can often be achieved in fewer plots, replaces (ii) by a combinatorial condition on the incidence matrix for treatments following each other.

Familiar combinatorial objects such as
doubly regular tournaments, 2-designs,
strongly regular graphs and S-digraphs can be used to
construct circular designs with weak neighbour balance.

Small example: each treatment comes "once" per block

Wind →

6	0	1	2	3	4	5	6
5	0	2	4	6	1	3	5
3	0	4	1	5	2	6	3
6	0	1	2	3	4	5	6
5	0	2	4	6	1	3	5
4	0	3	6	2	5	1	4
3	0	4	1	5	2	6	3
2	0	5	3	1	6	4	2
1	0	6	5	4	3	2	1

$s_{ij} :=$ # times i is directly upwind of j

$$S = \begin{array}{c} \\ 0\\1\\2\\3\\4\\5\\6 \end{array} \begin{array}{c} \begin{array}{ccccccc} 0&1&2&3&4&5&6 \end{array} \\ \left(\begin{array}{ccccccc} 0&2&2&1&2&1&1\\ 1&0&2&2&1&2&1\\ 1&1&0&2&2&1&2\\ 2&1&1&0&2&2&1\\ 1&2&1&1&0&2&2\\ 2&1&2&1&1&0&2\\ 2&2&1&2&1&1&0 \end{array}\right) \end{array}$$

Definitions of neighbour balance

$s_{ij} :=$ # times i is directly upwind of j

A design with t treatments each occurring once in each circular block of size t is

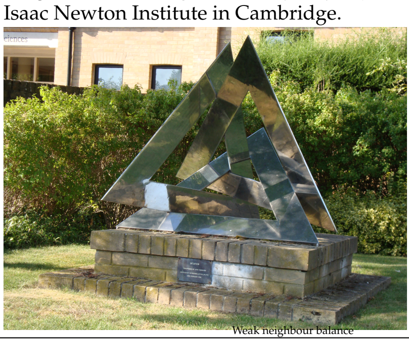
- strongly neighbour balanced if S is a multiple of the all-1 matrix J;
- neighbour balanced if S is a multiple of $J - I$;
- weakly neighbour balanced if
  - S has zero diagonal
  - and there is some λ such that $s_{ij} \in \{\lambda - 1, \lambda\}$ if $i \neq j$
  - and $S^\top S$ is completely symmetric (a linear combination of I and J).

The final condition occurs in the definition of many combinatorial objects.

Beginnings of weakly neighbour balanced designs

KF and AM defined WNBDs (weakly neighbour balanced designs) and found some by brute computer search.

They presented a talk about them during the six-month 2011 Programme on *Design and Analysis of Experiments* held at the Isaac Newton Institute in Cambridge.



RAB sketched out some ideas for a general method of construction.

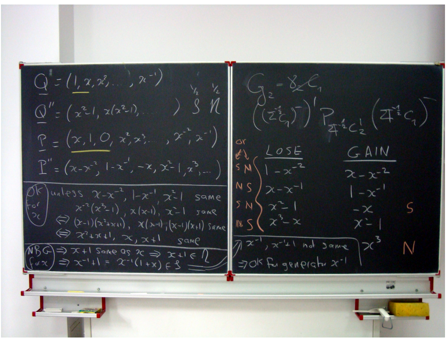
A workshop on neighbour balanced designs

KF organized a small research group meeting
(six people in one room with a blackboard)
on neighbour designs at Będlewo, Poland, in May 2013.

During this, KF, AM and JK showed that WNBDs are universally optimal (in a precise technical statistical sense).

Proof of one method of construction

During the workshop, RAB found a general method of construction when t is a prime power congruent to 3 modulo 4.



A 0, 1-matrix

s_{ij} := # times i is directly upwind of j

If we have a design which is weakly neighbour balanced but not neighbour balanced then S has zero diagonal, some other entries $\lambda - 1$ and some other entries λ. Put

$$A = S - (\lambda - 1)(J - I).$$

Then

- A is not zero;
- all entries of A are in $\{0, 1\}$;
- A has zero diagonal;
- A has constant row-sums and constant column-sums;
- $A^\top A - (\lambda - 1)(A + A^\top)$ is completely symmetric.

We know something about (some) matrices like this!

Three types

- A is not zero;
- all entries of A are in $\{0, 1\}$;
- A has zero diagonal;
- A has constant row-sums and constant column-sums;
- $A^\top A - (\lambda - 1)(A + A^\top)$ is completely symmetric.

We say that the design has

Type I if $A + A^\top$ is completely symmetric;
Type II if $A + A^\top$ is not completely symmetric and $\lambda = 1$;
Type III if $A + A^\top$ is not completely symmetric and $\lambda > 1$.

If Type I or Type II, then $A^\top A$ is completely symmetric, with constant row- and column-sums, so A can be regarded as the incidence matrix of a symmetric 2-design.
If Type I, then A has $(t - 1)/2$ non-zero entries in each row and column, and so $t \equiv 3$ mod 4.
If Type III, then $A^\top A$ is not completely symmetric.

Hooray for Type I

Theorem
If a WNBD is juxtaposed with a NBD and the result is a WNBD, then the starting WNBD either is a NBD or has Type I.

Number the positions in each block 1, 2, . . . , starting at the windy end.

Theorem
If a WNBD has the property that each numbered position has all treatments equally often, then it either is a NBD or has Type I.

Type I: $A + A^\top$ and $A^\top A$ are both completely symmetric

We can regard A as the adjacency matrix of a digraph Γ. The above conditions are equivalent to Γ being a doubly regular tournament. These are conjectured to exist whenever $t \equiv 3$ mod 4. If t is prime power we can put $A_{ij} = 1$ if and only if $j - i$ is a non-zero square in GF(t). If t is prime then

		$x \in Z_t$	$x + 1$	
$0 \neq y^2 \in Z_t$	. . .	xy^2	$(x+1)y^2$	. . .

is a WNBD.

$t = 3$ ✓, but too small to separate direct effects from upwind effects
$t = 7$ ✓, see next slide

Type I and $t = 7$: 3 blocks or 9 blocks

(Remember to loop each block into a circle!)

0	1	2	3	4	5	6
0	2	4	6	1	3	5
0	4	1	5	2	6	3

0	1	2	3	4	5	6
0	2	4	6	1	3	5
0	3	6	2	5	1	4
0	4	1	5	2	6	3
0	5	3	1	6	4	2
0	6	5	4	3	2	1

$t = 11$ ✓

$t = 15$?

RAB visited a different collaborator in the Poznań University of Life Sciences in July 2014.

KF asked "Why can't you do $t = 15$?"

RAB tried using A as the incidence matrix of PG$(3, 2)$ and proved that it is impossible.

$t = 15$: not finished yet

During the following weekend, RAB told PJC about this.

PJC said "You do know that there are other isomorphism classes of BIBDs for 15 points in 15 blocks of size 7, don't you?"

Type I and $t = 15$

Reid and Brown give the following doubling construction.

$$A_2 = \begin{pmatrix} A_1^\top & 0_t & A_1 + I_t \\ 1_t^\top & 0 & 0_t^\top \\ A_1 & 1_t & A_1 \end{pmatrix}$$

If A_1 is Type I for t then A_2 is Type I for $2t + 1$.

Doing this with $t = 7$ gives a doubly regular tournament Γ_2 on 15 vertices with an automorphism π of order 7.
If we can find a Hamiltonian cycle φ which has no edge in common with any of $\pi^i(\varphi)$ for $i = 1, \ldots, 6$, then $\varphi, \pi(\varphi), \ldots, \pi^6(\varphi)$ make a WNBD.

Annual meeting of the Portuguese Mathematical Society, in Lisboã, in the following week in July 2014

When the going got tough in the talks, RAB sat at the back and tried and failed to find such a Hamiltonian cycle φ by hand.

PJC used GAP, and found 120 solutions.

Question

$$\begin{array}{cccc} \text{treatments} & t & & 2t+1 \\ & & \text{doubling} & \\ \text{matrix} & A_1 & \longrightarrow & A_2 \\ & \updownarrow & & \updownarrow \\ \text{digraph} & \Gamma_1 & & \Gamma_2 \\ & \uparrow & & \uparrow \\ \text{design} & \Delta_1 & & \Delta_2 \end{array}$$

Could we go directly from Δ_1 to Δ_2?

Type I designs with rows and columns

Suppose that $t \equiv 3 \bmod 4$ and t is a prime power.
Let x be a primitive element of GF(t).
In the circular sequence

$$(1, x, x^2, x^3, \ldots, x^{t-1})$$

the successive differences give all non-zero elements of GF(t).

$$\text{Put } \phi = (x, 1, 0, x^2, x^3, \ldots, x^{t-1}).$$

If $t \neq 3$ then the number of non-zero squares in the successive differences of ϕ is one different from the number of non-squares in the successive differences of ϕ.

The $t(t-1)/2$ sequences $s\phi + i$, where s is a non-zero square in GF(t) and $i \in$ GF(t), give a weakly neighbour-balanced design in which every treatment occurs $(t-1)/2$ times in each numbered position.

That blackboard theorem

If ϕ is beautiful
(the number of non-zero squares in the successive differences of ϕ is one different from the number of non-squares in the successive differences of ϕ)
then that straightforward direct construction gives a WNBD for t treatments in $t(t-1)/2$ blocks of size t.

If the small thing is beautiful, then the big thing that I make from it has the properties that I want.

New Zealand, September 2014

PJC and RAB worked with collaborators at the University of Auckland on various other things. In our time off, we gave some more constructions and non-existence results.

Type II: $A^\top A$ is completely symmetric and $\lambda = 1$

Now we can regard A as the incidence matrix of a 2-design, with blocks labelled so that the diagonal is zero.

Using familiar tricks for constructing BIBDs (such as perfect difference sets), we can construct WNBDs.

We can also take advantage of symmetry to find a single Hamiltonian cycle whose images under a group of automorphisms of Γ give the blocks of the WNBD.

If A is itself symmetric then it is the adjacency matrix of a strongly regular graph in which every pair of distinct vertices have the same number of common neighbours (for example, the Shrikandhe graph and the Clebsch graph).

Again, familiar tricks and use of symmetry give us WNBDs.

Type II: an example with $t = 7$

In Z_7, the subset $\{2,4,5,6\}$ is a perfect difference set.

	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

$$S = A = \begin{array}{c} \\ 0\\1\\2\\3\\4\\5\\6 \end{array}\begin{array}{c} \begin{array}{ccccccc} 0&1&2&3&4&5&6 \end{array} \\ \begin{pmatrix} 0&0&1&0&1&1&1\\ 0&1&0&1&0&1&1\\ 1&1&0&0&1&0&1\\ 1&1&1&0&0&1&0\\ 0&1&1&1&0&0&1\\ 1&0&1&1&1&0&0\\ 0&1&0&1&1&1&0 \end{pmatrix} \end{array}$$

Type III: $A^\top A - (\lambda - 1)(A + A^\top)$ is completely symmetric, but $A^\top A$ and $(A + A^\top)$ are not

If A_1 has Type I for t treatments then

$$\begin{pmatrix} A_1 & A_1 + I_t & \dots & A_1 + I_t \\ A_1 + I_t & A_1 & \dots & A_1 + I_t \\ \vdots & \vdots & \ddots & \vdots \\ A_1 + I_t & A_1 + I_t & \dots & A_1 \end{pmatrix}$$

has Type III for mt treatments with $\lambda = m(t+1)/4$

and

$$\begin{pmatrix} 0 & 1_t^\top & 0 & 0_t^\top \\ 0_t & A_1 & 1_t & A_1^\top \\ 0 & 0_t^\top & 0 & 1_t^\top \\ 1_t & A_1^\top & 0_t & A_1 \end{pmatrix}$$

has Type III for $2(t+1)$ treatments with $\lambda = (t+1)/2$.

The second type is the adjacency matrix of what Babai and Cameron call an S-digraph.
$t = 3$ leads to the only Type III WNBDs ($t = 6$ and $t = 8$) found by KF and AM.

Type III doubling (or multiplying) constructions

Again, is there a way of going directly from the smaller design to the larger one?

References

- K. Filipiak and A. Markiewicz: On universal optimality of circular weakly neighbor balanced designs under an interference model. *Communications in Statistics, Theory and Methods* **41** (2012), 2356–2366.
- K. B. Reid and E. Brown: Doubly regular tournaments are equivalent to skew Hadamard matrices. *Journal of Combinatorial Theory Series A* **12** (1972), 332–338.
- L. Babai and P. J. Cameron: Automorphisms and enumeration of switching classes of tournaments. *Electronic Journal of Combinatorics* **7** (2000), R38.
- R. A. Bailey, P. J. Cameron, K. Filipiak, J. Kunert and A. Markiewicz: On optimality and construction of circular repeated-measurements designs. *Statistica Sinica* **27** (2017), 1–22.

All kinds of Mathematics . . .