Relations among partitions

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I have nice incidence relations between sets

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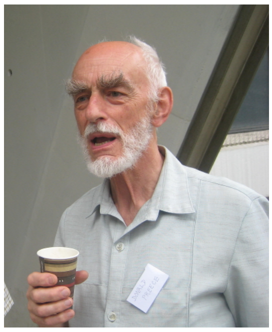


I have several factors in this experiment

An example of three uniform partitions of the same set

A	D	G
B	E	H
C	F	I

A	B	C
D	E	F
G	H	I

A	B	C
E	F	D
I	G	H

A	B	C
F	D	E
H	I	G

- The underlying set has size 36 (vegetable patches).
- The partition D into districts has 4 parts of size 9.
- The partition G into gardens has 12 parts of size 3.
- The partition L into letters (lettuce varieties) has 9 parts of size 4.

Three binary relations:

- $G \prec D$, G is a **refinement** of D;
- $L \perp\!\!\!\perp D$, L is **strictly orthogonal** to D;
- $L \rhd G$, L is **balanced** with respect to G.

What is a design?

A design is a collection of partitions on a (finite) set.

What are the relations among those partitions?

Uniform partitions

Definition
Let Ω be a finite set, and let F be a partition of Ω.
Then F is **uniform**
(or balanced
or homogeneous
or proper
or equireplicate
or regular)
if all parts of F have the same size.

Some definitions for a uniform partition of a finite set

Ω is the underlying set, of size e.
$V_0 =$ subspace of $\mathbb{R}^\Omega$ consisting of constant vectors.

For a given uniform partition F:

- $n_F =$ number of parts of F;
- $k_F =$ size of each part of F;
- $V_F =$ subspace of $\mathbb{R}^\Omega$ consisting of vectors which are constant on each part of F;
- $V_0 \leq V_F$ and $\dim(V_F) = n_F$;
- X_F is the $e \times n_F$ incidence matrix of elements of Ω in parts of F;
- $P_F = \dfrac{1}{k_F} X_F X_F^\top =$ matrix of orthogonal projection onto V_F, which averages each vector over each part of F.

Two partitions: incidence marix

Definition
If F and G are partitions of Ω then
the $n_F \times n_G$ **incidence matrix** N_{FG} is defined by

$$N_{FG} = X_F^\top X_G.$$

The entry in row i and column j is the size of the intersection of the i-th part of F with the j-th part of G.

Refinement

Definition
Let F and G be partitions of Ω.
Then F is **finer** than G (written $F \prec G$) if
each part of F is contained in a single part of G but $n_F > n_G$.

If $F \prec G$ then $V_G < V_F$.

If U is the partition with a single part, and E is the partition into singletons, then $E \preceq F \preceq U$ for all partitions F.

$$V_E = \mathbb{R}^\Omega \qquad V_U = V_0 = \text{subspace of constant vectors}$$

The relation $\preceq$ is a **partial order**.
The **infimum** $F \wedge G$ is the coarsest partition finer than, or equal to, both F and G. The **supremum** $F \vee G$ is the finest partition coarser than, or equal to, both F and G.

Theorem
$V_F \cap V_G = V_{F \vee G}$.

Strict orthogonality

$P_F =$ projector on V_F; $P_G =$ similar; $P_0 =$ projector on V_0.

$F(\omega) =$ the part of F containing element ω of Ω.

If F and G are partitions then V_F and V_G both contain V_0, so these spaces cannot be orthogonal to each other.

Definition
Partitions F and G are **strictly orthogonal** to each other (written $F \underline{\perp} G$) if $(V_F \cap V_0^\perp) \perp (V_G \cap V_0^\perp)$.

Equivalent conditions	Mode of thinking
$(V_F \cap V_0^\perp) \perp (V_G \cap V_0^\perp)$	angles between subspaces
$X_F^\top (I - P_0) X_G = 0$	matrix equation
$P_F P_G = P_G P_F = P_0$	matrix equation
$\frac{\lvert F \wedge G(\omega)\rvert}{\lvert\Omega\rvert} = \frac{\lvert F(\omega)\rvert}{\lvert\Omega\rvert} \times \frac{\lvert G(\omega)\rvert}{\lvert\Omega\rvert}$	a counting equation
	—proportional meeting

Orthogonal arrays

Definition
An **orthogonal array of strength two** is a collection of at least two uniform partitions on a finite set with the property that each pair is strictly orthogonal.

Example (11 partitions with 2 parts of size 6)

F_1	0	0	1	0	0	0	1	1	1	0	1	1
F_2	1	0	0	1	0	0	0	1	1	1	0	1
F_3	0	1	0	0	1	0	0	0	1	1	1	1
F_4	1	0	1	0	0	1	0	0	0	1	1	1
F_5	1	1	0	1	0	0	1	0	0	0	1	1
F_6	1	1	1	0	1	0	0	1	0	0	0	1
F_7	0	1	1	1	0	1	0	0	1	0	0	1
F_8	0	0	1	1	1	0	1	0	0	1	0	1
F_9	0	0	0	1	1	1	0	1	0	0	1	1
F_{10}	1	0	0	0	1	1	1	0	1	0	0	1
F_{11}	0	1	0	0	0	1	1	1	0	1	0	1

General orthogonality

In general, $V_F \cap V_G = V_{F \vee G}$,
where $F \vee G$ is the supremum of F and G.

Definition
Partitions F and G are **orthogonal** to each other (written $F \perp G$) if $(V_F \cap V_{F \vee G}^\perp) \perp (V_G \cap V_{F \vee G}^\perp)$.

Equivalently,

- $X_F^\top (I - P_{F \vee G}) X_G = 0$;
- $P_F P_G = P_G P_F$ (projectors commute);
- proportional meeting within each part of $F \vee G$.

If $F \preceq G$ then $F \perp G$.

In particular, $F \perp F$ for all partitions F.

Why orthogonal projection? Back to gardening experiment

A	D	G	A	B	C	A	B	C	A	B	C
B	E	H	D	E	F	E	F	D	F	D	E
C	F	I	G	H	I	I	G	H	H	I	G

- 12 gardens, each containing three vegetable patches;
- 9 lettuce varieties, each grown on four patches.

Denote by Y_ω the total yield of edible lettuce on patch ω.
Assume that Y_ω is a random variable with expectation

$$\tau_{L(\omega)} + \beta_{G(\omega)}.$$

I could add 51 to each τ_i and subtract 51 from each β_j without changing this. I would like to estimate $\tau_1, \ldots, \tau_9$ up to an additive constant. I do not care about $\beta_1, \ldots, \beta_{12}$.
So I put the responses Y_ω into a vector $\mathbf{Y}$ and project it onto $V_G^\perp$.
There are no β_j in the expectation of $(I - P_G)\mathbf{Y}$.

Orthogonality: recap

$P_0 =$ projector onto space V_0 of constant vectors.

$F \perp\!\!\!\perp G$ means that $(I - P_0)(V_F) \perp (I - P_0)(V_G)$;
equivalently, $X_F^\top (I - P_0) X_G = 0$.

We could say that F is orthogonal to G after adjusting for the partition U with a single part.

$F \perp G$ means that $(I - P_{F\vee G})(V_F) \perp (I - P_{F\vee G})(V_G)$;
equivalently, $X_F^\top (I - P_{F\vee G}) X_G = 0$.

We could say that F is orthogonal to G after adjusting for their supremum $F \vee G$.

Adjusted orthogonality

$N_{RC} = X_R^\top X_C = n_R \times n_C$ incidence matrix of R-parts with C-parts

Definition
Let R (rows), C (columns) and L (letters) be three partitions on a finite set Ω. Then R and C have **adjusted orthogonality** with respect to L if $(I - P_L)(V_R) \perp (I - P_L)(V_C)$.

Equivalent conditions	Mode of thinking
$(I - P_L)(V_R) \perp (I - P_L)(V_C)$	angles between subspaces
$X_R^\top (I - P_L) X_C = 0$	matrix equation
$X_R^\top X_C = X_R^\top P_L X_C$	matrix equation
if L is uniform, $k_L N_{RC} = N_{RL} N_{LC}$	counting equation

The number of letters in common to row i and column j is

$$k_L \times |\text{row } i \cap \text{column } j|\,.$$

A nasty example of adjusted orthogonality (Preece, 1988)

A F	D	G	J	C
D	B G	E	H	F
G	E	C H	A	I
J	H	A	D I	B
C	F	I	B	E J

- $|\Omega| = 30$;
- 5 rows, each of size 6;
- 5 columns, each of size 6;
- 10 letters, each of “size” 3.

The number of letters in common to row i and column j is

$$\begin{cases} 6 & \text{if } i = j \\ 3 & \text{otherwise.} \end{cases}$$

A nicer example of adjusted orthogonality

H	J	I	G	F	E
J	I	H	C	B	D
D	F	A	J	G	C
A	B	G	E	D	I
E	A	C	B	H	F

- $|\Omega| = 30$;
- 5 rows, each of size 6;
- 6 columns, each of size 5;
- 10 letters, each of “size” 3.

The number of letters in common to row i and column j is always 3.

This is a consequence of adjusted orthogonality if $R \perp\!\!\!\perp C$ and $R \wedge C$ is uniform.

Some comments

Agrawal (1966) and Preece (1966) introduced the general idea of adjusted orthogonality in contemporaneous papers, but they neither defined it nor named it.

It had been previously used in isolated examples.

It was introduced, but not named, independently by several authors in the next decade.

Eccleston and Russell (1975) independently introduced the concept; they named it in a 1977 paper.

It took a while before *adjusted orthogonality* became the standard wording, so my survey may have missed some references.

Part of the difficulty may have been the three modes of thinking (angles between subspaces; matrix equations; a counting equation) about equivalent versions of the definition. Some results are obvious in one mode of thinking but not in the others.

A more general version of adjusted orthogonality

Eccleston and Russell (1975) actually proposed this more general definition.

Definition
Let $\mathcal{L}$ be a set of partitions of Ω. Put

$$V_{\mathcal{L}} = \sum_{L \in \mathcal{L}} V_L$$

and let $P_{\mathcal{L}}$ be the matrix of orthogonal projection onto $V_{\mathcal{L}}$. Then R and C have adjusted orthogonality with respect to $\mathcal{L}$ if

$$X_R^\top (I - P_{\mathcal{L}}) X_C = 0.$$

What about adjusted uniformity?

$$\boxed{N_{FG} = X_F^\top X_G}$$

We have seen how the relationship between two partitions is modified if we project onto the orthogonal complement of the subspace corresponding to a third partition. But what happens to properties of a single partition when we project like this?

Partition L is uniform means that
$X_L^\top X_L = N_{LL} =$ diagonal matrix of sizes of parts of L
is a (non-zero) multiple of the identity matrix I of order n_L.
This is a special case of a completely symmetric matrix (a linear combination of I and the all-1 matrix J).

So to say that L has adjusted uniformity with respect to partition B should mean that
$X_L^\top(I - P_B)X_L$ is completely symmetric but not zero.

But $X_L^\top(I - P_B)X_L = N_{LL} - \frac{1}{k_B}N_{LB}N_{BL}$: you might recognise this condition.

Balance

Definition
Let L and B be uniform partitions of Ω. Then L is **balanced** with respect to B if $X_L^\top(I - P_B)X_L = N_{LL} - \frac{1}{k_B}N_{LB}N_{BL}$ is completely symmetric but not zero.

- 'Non-zero' excludes $B \preceq L$ but does not exclude $B \perp L$.
- The (i,j)-entry of $N_{LB}N_{BL}$ is the number of times that letters i and j concur in blocks (allowing for multiplicities).
- Statisticians always call this property 'balance', but some of you may say that the parts of L and B form the points and blocks of a 2-design.

Definition
The relationship between L and B is **binary** if all parts of $L \wedge B$ are singletons; it is **generalized binary** if no pair of parts of $L \wedge B$ have sizes differing by more than one.

Balance, continued

Write $L \blacktriangleright B$ if L is balanced with respect to B but L is not strictly orthogonal to B.

Write $L \rhd B$ if $L \blacktriangleright B$ and the relationship beween L and B is (generalized) binary.

Write $L \bowtie B$ if $L \rhd B$ and $B \rhd L$.
If the relationship between L and B is binary and $L \bowtie B$, then we have a **symmetric balanced incomplete-block design**.

A	D	G	A	B	C	A	B	C	A	B	C
B	E	H	D	E	F	E	F	D	F	D	E
C	F	I	G	H	I	I	G	H	H	I	G

In the gardening experiment,
$L \rhd G$ and the relationship is binary.

Two block designs in which letters are balanced with respect to blocks, which are represented by columns

(a)

A	B	C	D	E	F	G
A	A	A	A	A	A	A
B	B	B	B	B	B	B
C	C	C	C	C	C	C
D	D	D	D	D	D	D
E	E	E	E	E	E	E
F	F	F	F	F	F	F
G	G	G	G	G	G	G

(b)

A	B	C	D	E	F	G
A	B	C	D	E	F	G
C	D	E	F	G	A	B
C	D	E	F	G	A	B
D	E	F	G	A	B	C
D	E	F	G	A	B	C
E	F	G	A	B	C	D
E	F	G	A	B	C	D

(a) is generalized binary but not binary;
(b) is not generalized binary.

A more general version of balance

Definition
Let $\mathcal{G}$ be a set of partitions of Ω. Put

$$V_{\mathcal{G}} = \sum_{G \in \mathcal{G}} V_G$$

and let $P_{\mathcal{G}}$ be the matrix of orthogonal projection onto $V_{\mathcal{G}}$.
Then L is balanced with respect to $\mathcal{G}$ if

$$X_L^\top(I - P_{\mathcal{G}})X_L \text{ is completely symmetric but not zero.}$$

What about three partitions? Or more?

Let R, C and L be uniform partitions of Ω.

If all three pairwise relations are orthogonality (possibly including refinement) then we get a nice decomposition of $\mathbb{R}^\Omega$ into orthogonal subspaces, and each pair has adjusted orthogonality with respect to the third.

Suppose that $R \perp C$, $R \perp L$ and $L \rhd C$.

- Projecting onto $V_R^\perp$ leaves $V_C \cap V_0^\perp$ and $V_L \cap V_0^\perp$ unchanged, so the relation between L and C is unchanged.
- Projecting onto $V_L^\perp$ leaves $V_R \cap V_0^\perp$ unchanged and leaves $V_C \cap V_0^\perp$ inside $V_L + V_C$, which is orthogonal to $V_R \cap V_0^\perp$, so R and C have adjusted orthogonality with respect to L.

More generally, given a set $\mathcal{F}$ of partitions, if each F in $\mathcal{F}$ is non-orthogonal to at most one of the others then the pairwise relations suffice to describe the system.

Youden squares

Definition (Youden, 1937)

An $n \times m$ **Youden square** is a set of size nm with uniform partitions into n rows (R), m columns (C) and m letters (L) such that all pairwise relations are binary, $R \perp C$, $R \perp L$ and $L \bowtie C$.

Example ($n = 3$ and $m = 7$)

A	B	C	D	E	F	G
B	D	F	E	G	A	C
C	F	E	A	B	G	D

Theorem

Every symmetric balanced incomplete-block design can be arranged as a Youden square.

Proof.

Use Hall's Marriage Theorem to sequentially choose the letters in each row as a set of distinct representatives. □

Double Youden rectangles

Definition (Bailey, 1989)

An $n \times m$ **double Youden rectangle** is a set of size nm with uniform partitions into n rows (R), m columns (C), m Latin letters (L) and n Greek letters (G) such that all pairwise relations (apart from that between R and G) are binary, $R \perp C$, $R \perp L$, $G \perp C$, $G \perp L$, $L \bowtie C$ and $R \bowtie G$.

Example ($n = 4$ and $m = 13$, Preece (1982))

A ♠	3 ♣	4 ♡	7 ♡	8 ♣	2 ♣	10 ♢	J ♠	5 ♠	6 ♢	Q ♢	K ♠	9 ♡
2 ♢	5 ♡	3 ♢	4 ♠	6 ♠	7 ♢	8 ♠	9 ♣	10 ♣	K ♡	J ♡	Q ♣	A ♢
4 ♣	J ♢	6 ♣	K ♢	5 ♢	9 ♠	7 ♣	8 ♡	Q ♡	10 ♠	A ♣	2 ♡	3 ♠
10 ♡	2 ♠	Q ♠	5 ♣	A ♡	6 ♡	3 ♡	4 ♢	9 ♢	J ♣	7 ♠	8 ♢	K ♣

Triple arrays

Definition (McSorley, Phillips, Wallis and Yucas, 2005)

An $r \times c$ rectangle with one of v letters allocated to each cell is an **triple array** if all partitions are uniform, all pairwise relations are binary, $R \perp C$, $R \rhd L$, $C \rhd L$ and R and C have adjusted orthogonality with respect to L.

So $n_R = r = k_C$, $n_C = c = k_R$, $n_L = v$ and $k_L = rc/v$.

Also, every pair of rows have the same number of letters in common,
every pair of columns have the same number of letters in common,
and every row has k_L letters in common with every column.

These are among the designs discussed by Preece (1966) and Agrawal (1966).

Extremal triple arrays

Theorem (Bagchi, 1998)

If a triple array has r rows, c columns and v letters then $v \geq r + c - 1$.

Definition

A triple array is **extremal** if $v = r + c - 1$.

Given an extremal triple array, the following construction gives a symmetric balanced incomplete-block design (SBIBD) for $r + c$ points in blocks of size r.

1. The points are the (names of the) rows and columns.
2. Each letter gives a block, consisting of the columns in which it occurs and the rows in which it does not occur.
3. The final block contains (the names of) all the rows.

An extremal triple array with $r = 5$, $c = 6$ and $v = 10$

	0	2	6	7	8	X
1	B	A	E	D	J	F
4	G	H	B	I	D	E
9	J	I	A	B	C	G
5	F	J	H	C	E	I
3	H	D	C	F	G	A

An $r \times c$ rectangle, each cell containing one of $r + c - 1$ letters, such that

- rows R are strictly orthogonal to columns C, with all intersections of size 1;
- rows are balanced with respect to letters (L) (every pair of rows has the same number of letters in common);
- columns are balanced with respect to letters;
- rows and columns have adjusted orthogonality with respect to L (the set of letters in each row has constant size of intersection with the set of letters in each column).

Triple array to SBIBD

	0	2	6	7	8	X
1	B	A	E	D	J	F
4	G	H	B	I	D	E
9	J	I	A	B	C	G
5	F	J	H	C	E	I
3	H	D	C	F	G	A

- The points are 1, 4, 9, 5, 3, 0, 2, 6, 7, 8, X.
- Block A contains points 2, 6, X, 4, 5.
- And so on.
- Block J contains points 0, 2, 8, 4, 3.
- The final block contains points 1, 4, 9, 5, 3.

Start with a SBIBD: can we construct the triple array?

	A	B	C	D	E	F	G	H	I	J
1	2	3	4	5	6	7	8	9	X	0
4	5	6	7	8	9	X	0	1	2	3
9	X	0	1	2	3	4	5	6	7	8
5	6	7	8	9	X	0	1	2	3	4
3	4	5	6	7	8	9	X	0	1	2

	0	2	6	7	8	X	row name is not in
1				BDF			A B D E F J
4							B D E G H I
9							A B C G I J
5							C E F H I J
3							A C D F F H
column name is in	B F G H J	A D H I J	A B C E H	B C D F I	C D E G J	A E F G I	

Put one letter in each cell and obtain these subsets in rows and columns

Problem: can you do it?

Given a subset of letters allowed for each cell,
is it possible to choose an **array of distinct representatives**,
one per cell, so that no letter is repeated in a row or column?

Fon-der-Flaass, 1997: the general problem is NP-complete.

Suppose the allowable subsets come from an SBIBD in the way that I showed?

- Not if the allowable subsets have size ≤ 2.
- Agrawal (1966): if $k_L > 2$ then it was "always possible in the examples tried by the author".
- Rhagavarao and Nageswararao (1974): two false proofs.
- Seberry (1979); Street (1981); Bailey and Heidtmann (1994); Bagchi (1998); Preece, Wallis and Yucas (2005) gave explicit constructions for $q \times (q+1)$ when q is an odd prime power and $q > 3$.
- Computer search always gives a positive result if $k_L > 2$.

Your task: Proof or counter-example.

Look at balance again

P_F = matrix of orthogonal projection onto V_F

P_0 = matrix of orthogonal projection onto V_0

Put $Q_F = P_F - P_0$.

F is balanced with respect to G means that
$N_{FG}N_{GF}$ is completely symmetric but not scalar; equivalently
$X_F^\top(I - P_G)X_F$ is completely symmetric but not zero.

If we want to exclude strict orthogonality, then the condition becomes
$X_F^\top(I - P_G)X_F$ is completely symmetric but not a multiple of J.

Equivalently, there is a scalar μ with $0 < \mu < 1$ such that
$Q_F Q_G Q_F = \mu Q_F$.

Balance among three or more uniform partitions

If $\mathcal{G}$ is a set of partitions of Ω,

$$P_{\mathcal{G}} = \text{matrix of orthogonal projection onto } \sum_{G \in \mathcal{G}} V_G.$$

F is **balanced with respect to $\mathcal{G}$** if
$X_F^\top(I - P_{\mathcal{G}})X_F$ is completely symmetric but not zero.

To exclude orthogonality, require that
$X_F^\top(I - P_{\mathcal{G}})X_F$ is completely symmetric but not a multiple of J.

Equivalently, there is a scalar μ with $0 < \mu < 1$ such that
$Q_F Q_{\mathcal{G}} Q_F = \mu Q_F$,
where $Q_{\mathcal{G}} = P_{\mathcal{G}} - P_0$.

(Statements in the remaining slides may not be consistent about this exclusion.)

Exactly three partitions

Suppose that partitions F, G and H each have n parts of size k, and that each pair are balanced (both ways).

Then F is balanced with respect to $\{G, H\}$ if and only if

$$N_{FG}N_{GH}N_{HF} + N_{FH}N_{HG}N_{GF} \text{ is completely symmetric.}$$

Equivalently,

$$Q_F(Q_G Q_H + Q_H Q_G)Q_F \text{ is a non-zero multiple of } Q_F.$$

The above is implied by this stronger condition:

$$N_{FG}N_{GH} \text{ is a linear combination of } N_{FH} \text{ and } J.$$

My attempt at a general definition

A set $\mathcal{F}$ of uniform partitions of Ω, all with n parts,
has **universal balance** if
whenever $F \in \mathcal{F}$ and $\mathcal{L} \subseteq \mathcal{F} \setminus \{F\}$
then F is balanced with respect to $\mathcal{L}$
but $V_F \cap V_0^\perp$ is not orthogonal to $V_{\mathcal{L}} \cap V_0^\perp$.

Equivalently, whenever F and $\mathcal{L}$ are as above, then there is a scalar μ with $0 < \mu < 1$ such that $Q_F Q_{\mathcal{L}} Q_F = \mu Q_F$.

Matrix conditions for universal balance

Theorem
If $\mathcal{F}$ has universal balance and $\mathcal{L} \subseteq \mathcal{F}$ then $Q_{\mathcal{L}}$ is a linear combination of products of the matrices Q_L for L in $\mathcal{L}$.

Corollary
If $\mathcal{F}$ has universal balance and $\mathcal{L} \subset \mathcal{F}$ and $F \in \mathcal{F} \setminus \mathcal{L}$ then $X_F^{\perp} Q_{\mathcal{L}} X_F$ is a sum of matrices of the form

$$N_{FL_1} N_{L_1L_2} \cdots N_{L_rF} \qquad (1)$$

where $(L_1, L_2, \ldots, L_r)$ is a sequence of partitions in $\mathcal{L}$, possibly having repeated entries.

So, if we can ensure that, whenever M is a product like (1) then $M + M^{\perp}$ is completely symmetric, then we have universal balance.

Known families, for n parts of size k

$$N_{FG}N_{GH}N_{HF} + N_{FH}N_{HG}N_{GF} \text{ is completely symmetric,}$$

or its generalization.

- $k = n - 1$: remove a common transversal from a set of mutually orthogonal $n \times n$ Latin squares, so that every N is $J - I$. (Done by many people.)
- $n \equiv 3 \pmod 4$ and $k = (n+1)/2$ or $k = (n-1)/2$: if there is a doubly-regular tournament of size n, its adjacency matrix A satisfies $I + A + A^{\top} = J$ and $A^2 \in \langle I, A, J \rangle$, then ensure that each N is either $I + A$ or $I + A^{\top}$ (or A or $A^{\top}$). (Done by many people, usually without using the words *doubly regular tournament*.)

Known families, for n parts of size k, continued

$$N_{FG}N_{GH}N_{HF} + N_{FH}N_{HG}N_{GF} \text{ is completely symmetric,}$$

or its generalization.

- $n = 2^{2m}$ and $k = 2^{2m-1} + 2^{m-1}$ or $k = 2^{2m-1} - 2^{m-1}$: Cameron and Seidel (1973) have constructions from quadratic forms, and the strong form of the condition is satisfied. (For $n = 16$ and $k = 6$ this involves compatible Clebsch graphs which form an amorphic association scheme.)

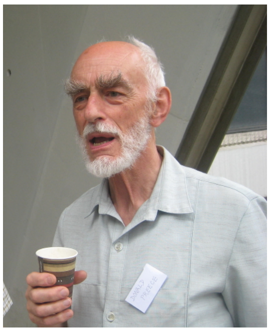
Problem: is this all?

Your task

- Find all possible sets of three or more incidence matrices N_{FG} satisfying the conditions.
- For each such set, realise them as incidence matrices of a set of partitions with n parts of size k.
- For each such realisation, find another partition with k parts of size n that is orthogonal to all the rest (surprisingly, this often makes the previous part easier).
- What about two such sets, one with n parts of size k, the other with k parts of size n, and every partition in one set orthogonal to every partition in the other set? (If each set has two partitions, this is a double Youden rectangle, so I only require one of the sets to have at least three partitions.)
- Or three or more?

Looking back at the Aberystwyth BCC in 1973

Ooh!—I know some suitable incidence matrices for those numbers



I want universal balance among some partitions with 16 parts of size 6

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P
A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P
A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P
H	G	F	E	D	C	B	A	P	O	N	M	L	K	J	I
G	H	E	F	C	D	A	B	O	P	M	N	K	L	I	J
B	A	D	C	F	E	H	G	J	I	L	K	N	M	P	O
K	L	I	J	O	P	M	N	C	D	A	B	G	H	E	F
J	I	L	K	N	M	P	O	B	A	D	C	F	E	H	G
D	C	B	A	H	G	F	E	L	K	J	I	P	O	N	M
M	N	O	P	I	J	K	L	E	F	G	H	A	B	C	D
I	J	K	L	M	N	O	P	A	B	C	D	E	F	G	H
E	F	G	H	A	B	C	D	M	N	O	P	I	J	K	L
O	P	M	N	K	L	I	J	G	H	E	F	C	D	A	B
F	E	H	G	B	A	D	C	N	M	P	O	J	I	L	K
L	K	J	I	P	O	N	M	D	C	B	A	H	G	F	E
P	O	N	M	L	K	J	I	H	G	F	E	D	C	B	A
C	D	A	B	G	H	E	F	K	L	I	J	O	P	M	N
N	M	P	O	J	I	L	K	F	E	H	G	B	A	D	C

Preceding slide, from Preece and Cameron (1975)

Underlying set has size 96.

16 columns of size 6.
16 top letters of size 6.
16 middle letters of size 6.
16 bottom letters of size 6.

Universal balance among the above,
which are all strictly orthogonal to:
6 rows of size 16.

Cameron says that he did not really understand this way of thinking about relations between partitions on a set until 25 years later, when he generalized this construction to arbitrary powers of 4 at the 2001 BCC in Sussex (Cameron, 2003).

Multi-layered Youden rectangles

- Each stage has m parts of size n.
- The set of stages has universal balance.
- Each layer has n parts of size m.
- The set of layers has universal balance.
- Every layer is strictly orthogonal to every stage.

Preece and Morgan (2017) introduced this name,
with the number of stages restricted to 2;
they gave some constructions
and proved some results.

Your task Keep going!