Latin cubes

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What is a Latin square?

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Let n be a positive integer.
A Latin square of order n is an $n \times n$ array of cells in which n symbols are placed, one per cell, in such a way that each symbol occurs once in each row and once in each column.

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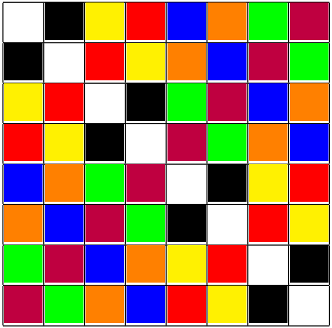
A Latin square of order 8



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Example

If Ω is the set of cells in a Latin square, then there are five natural uniform partitions of Ω:

- R each part is a row;
- C each part is a column;
- L each part consists of the those cells with a given letter;
- U the universal partition, with a single part;
- E the equality partition, whose parts are singletons.

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Definition

The supremum, or join, of partitions P and Q is the partition $P \vee Q$ which satisfies $P \preccurlyeq P \vee Q$ and $Q \preccurlyeq P \vee Q$ and if $P \preccurlyeq S$ and $Q \preccurlyeq S$ then $P \vee Q \preccurlyeq S$.

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Draw a graph by putting an edge between two points if they are in the same part of P or the same part of Q. Then the parts of $P \vee Q$ are the connected components of the graph.

Hasse diagrams

Given a collection $\mathcal{P}$ of partitions of a set Ω,
we can show them on a Hasse diagram.

- Draw a dot for each partition in $\mathcal{P}$.
- If $P \prec Q$ then put Q higher than P in the diagram.
- If $P \prec Q$ but there is no S in $\mathcal{P}$ with $P \prec S \prec Q$ then draw a line from P to Q.

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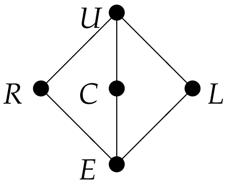
Here is the Hasse diagram for a Latin square.



An alternative definition of Latin square

Definition

Let P and Q be uniform partitions of a set Ω. Then P and Q are compatible if

- whenever ω_1 and ω_2 are points in the same part of $P \vee Q$, there are points α and β such that
  - ω_1 and α are in the same part of P,
  - α and ω_2 are in the same part of Q,
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- $P \wedge Q$ is uniform.

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Definition

A Latin square is a set $\{R, C, L\}$ of pairwise compatible uniform partitions of a set Ω which satisfy $R \wedge C = R \wedge L = C \wedge L = E$ and $R \vee C = R \vee L = C \vee L = U$.

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Comment

These definitions can be applied to finite or infinite sets.

Another nice family of partitions

Definition

Suppose that P_1, P_2 and P_3 are partitions of a set Ω, none of which is U. Then $\{P_1, P_2, P_3\}$ is a Cartesian decomposition of Ω of dimension 3 if $|\Gamma_1 \cap \Gamma_2 \cap \Gamma_3| = 1$ whenever Γ_i is a part of P_i for $i = 1, 2, 3$.

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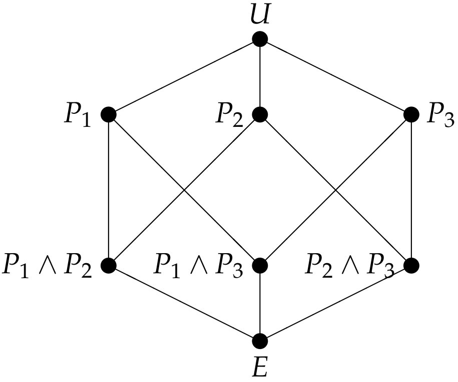
Taking infima gives a Cartesian lattice.



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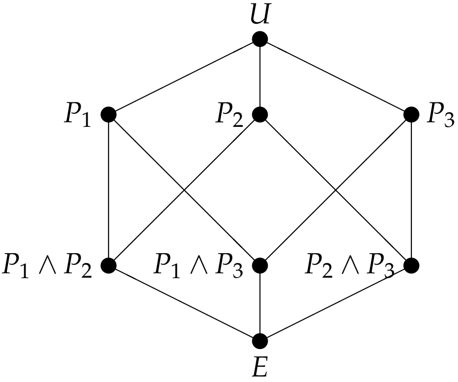

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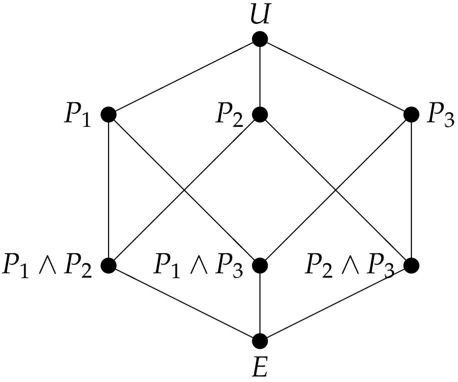

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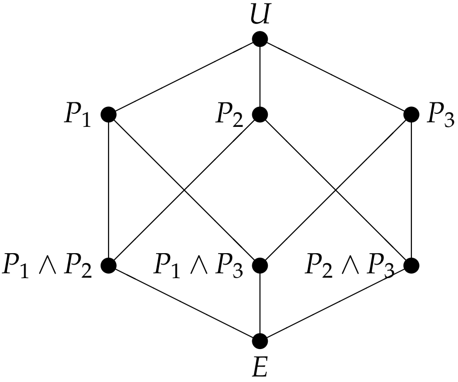

- Each partition is uniform.
- Each pair are compatible.
- Statisticians call this a completely crossed orthogonal block structure.

Coset partitions

Definition
Let H be a subgroup of a group G. Then P_H is the partition of G into right cosets of H.

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Proposition

Let H and K be subgroups of a group G. The following hold.

1. P_H *is uniform.*
2. $P_H \wedge P_K = P_{H \cap K}$.
3. $P_H \vee P_K = P_{\langle H, K \rangle}$.
4. P_H *and* P_K *are compatible if and only if* $HK = KH$.

Latin squares and quasigroups

If the rows, columns and letters of a Latin square are all labelled by the elements of the same set T, then the Latin square induces a quasigroup structure on T by the rule that $x \circ y = z$ if z is the letter in the cell in row x and column y.

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E	A	B	C	D
B	C	D	E	A
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Cayley table of cyclic group C_5

A	B	C	D	E
B	A	D	E	C
D	C	E	A	B
C	E	A	B	D
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Not a Cayley table of a group

The Quadrangle Criterion

Definition

A Latin square satisfies the quadrangle criterion if, whenever there are 2×2 subsquares

$$\begin{array}{|c|c|}\hline A & B \\ \hline C & X \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|c|c|}\hline A & B \\ \hline C & Y \\ \hline \end{array}$$

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So this combinatorial condition enables us to recognise a group: the algebra drops out of the combinatorics.

What is a (finite) Latin cube?

The structure of a cube is a Cartesian decomposition $\{P_1, P_2, P_3\}$ of a set Ω of dimension three, where P_i has n parts for $i \in \{1, 2, 3\}$.
Alternatively, $\Omega = \{(x, y, z) : x, y, z \in \{1, 2, \ldots, n\}\}$.

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Call the parts of P_1, P_2 and P_3 layers, and the parts of $P_1 \wedge P_2$, $P_1 \wedge P_3$ and $P_2 \wedge P_3$ lines. Two lines are parallel if they are parts of the same partition.

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There are three possibilities for allocating letters to make a Latin cube, giving a partition L into letters.

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(LC0) There are n letters, each of which occurs once per line.

(LC1) There are n letters, each of which occurs n times per layer.

(LC2) There are n^2 letters, each of which occurs once per layer.

We concentrate on Latin cubes of sort (LC2)

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This means that, for $i \in \{1, 2, 3\}$,
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Definition

A Latin cube of sort (LC2) is regular if, whenever Γ_1 and Γ_2 are parallel lines in the cube, the set of letters occurring in Γ_1 is either exactly the same as the set of letters occurring in Γ_2 or disjoint from it.

A Latin cube of sort (LC2) with $n = 3$ which is not regular

Horizontal layers are shown side by side.

A	E	F
H	I	D
C	G	B

D	B	I
E	C	G
F	A	H

G	H	C
B	F	A
I	D	E

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I	D	E

These two *yz*-lines have letter sets which are neither the same nor disjoint.

A Latin cube of sort (LC2) with $n = 3$ which is regular

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I	C	F
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These two *yz*-lines have identical letter sets; the other two *yz*-lines in the middle layer cannot have any of these letters.

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Horizontal layers are shown side by side.

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Horizontal layers are shown side by side.

A	*D*	*G*
I	*C*	*F*
E	*H*	*B*

B	*E*	*H*
G	*A*	*D*
F	*I*	*C*

C	*F*	*I*
H	*B*	*E*
D	*G*	*A*

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One supremum

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Let's make a part of L^{23}.

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Let's make a part of L^{23}. Start in one cell.

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A	D	G
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Let's make a part of L^{23}. Start in one cell.

Include everything else in the same part of $P_2 \wedge P_3$.

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Now the set of cells with red letters is a union of parts of $P_2 \wedge P_3$ as well as a union of parts of L, so it is a part of L^{23}.

One supremum

For $\{i,j,k\} = \{1,2,3\}$, put $L^{ij} = (P_i \wedge P_j) \vee L$.

A	*D*	*G*
I	*C*	*F*
E	*H*	*B*

B	*E*	*H*
G	*A*	*D*
F	*I*	*C*

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The cube has many of these, and the full details of the later proof (not shown) examine these in detail.

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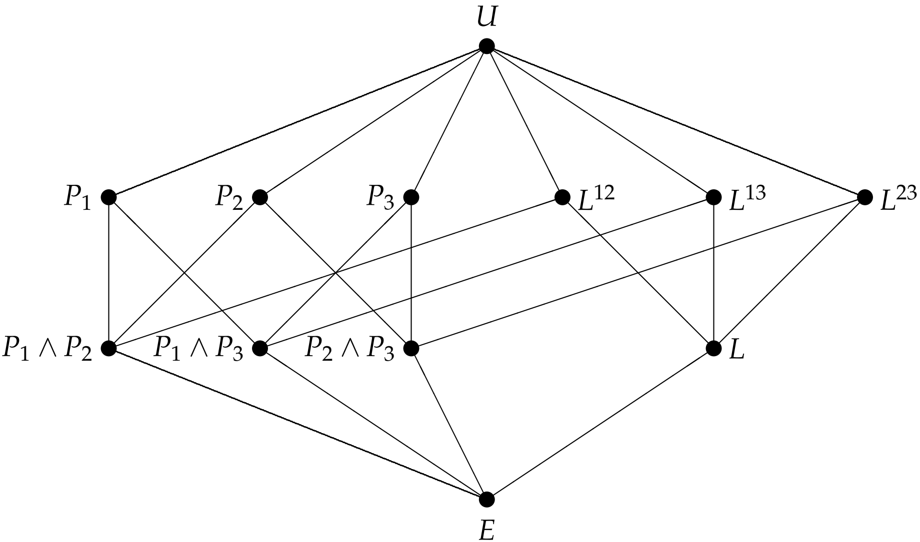
Theorem

If a Latin cube of sort (LC2) is regular then $\{P_3, L^{13}, L^{23}\}$ is a three-dimensional Cartesian decomposition of the cube. Moreover, $L^{13} \wedge L^{23} = L$, $P_3 \wedge L^{23} = P_2 \wedge P_3$ and $P_3 \wedge L^{13} = P_1 \wedge P_3$.

Hasse diagram for partitions discussed



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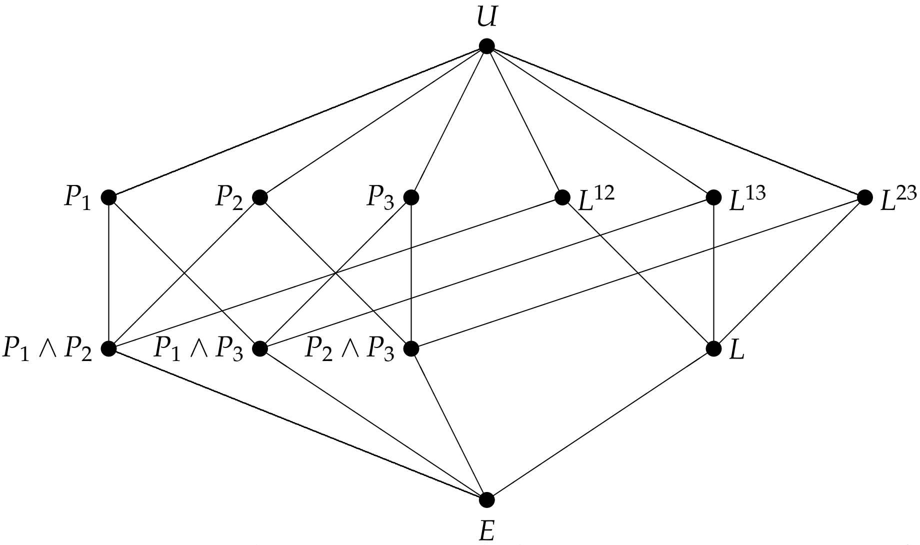

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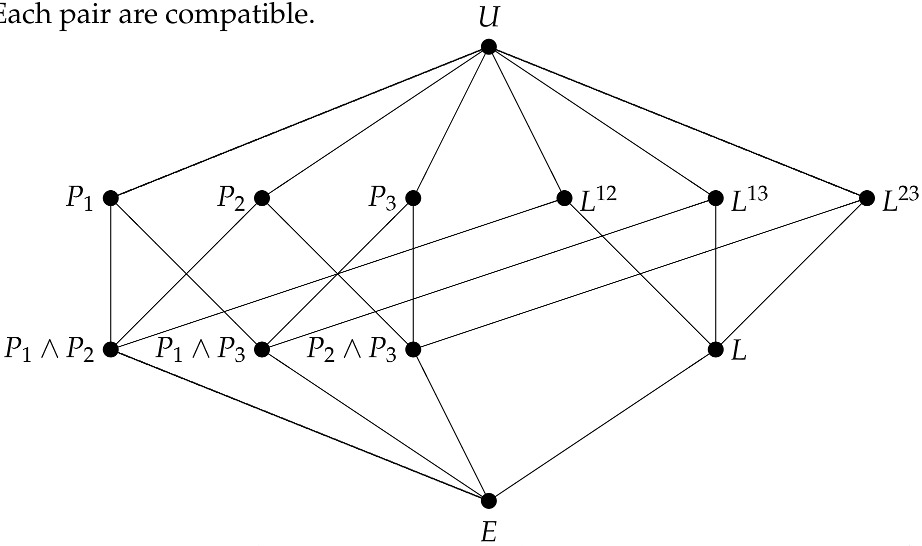

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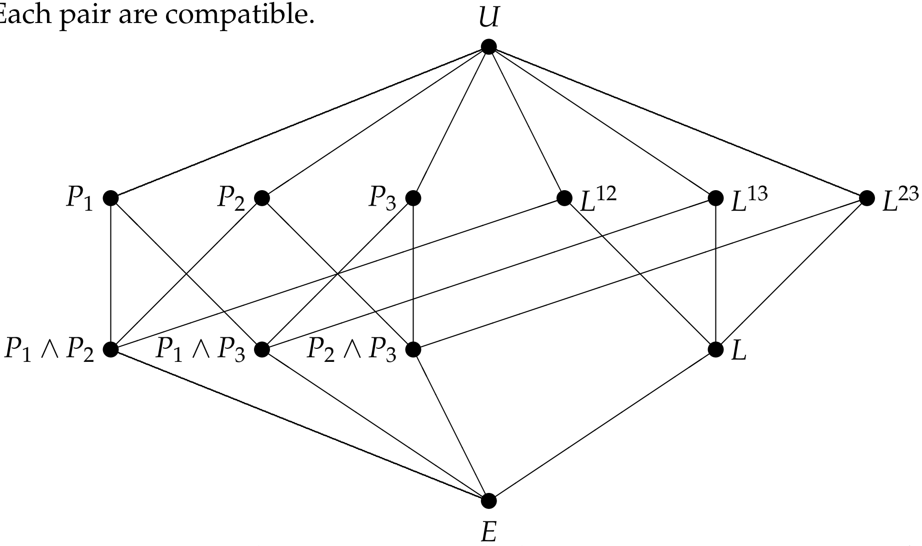

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Each partition is uniform.

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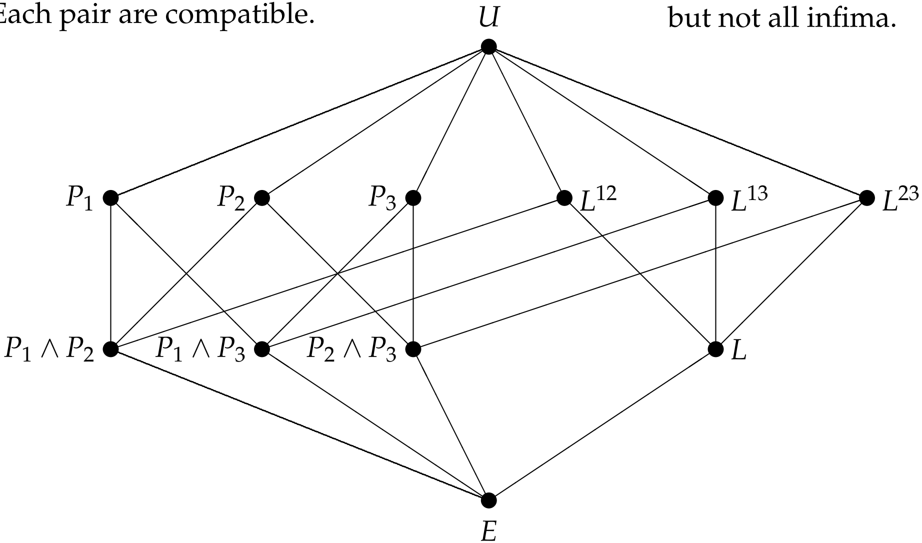

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(If we know $x^{-1}y$ and $x^{-1}z$ then we know $y^{-1}z$,
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$K_{12} \cap H_3 = \{(x,x,1) : x \in G\}$; $\quad K_{23} \cap H_1 = \{(1,y,y) : y \in G\}$.
If G is not abelian then these subgroups do not commute, so the partitions are not compatible, so we do not include all infima.

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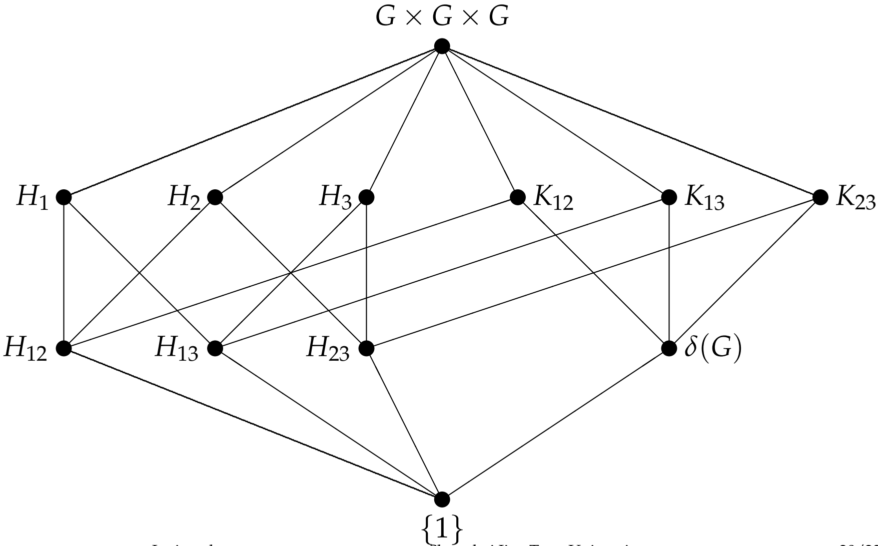

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Suppose that $\mathcal{Q} = \{Q_1, Q_2, Q_3, Q_4\}$ is a set of four partitions of the same set Ω. The following are equivalent.

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So, either of these two combinatorial conditions leads us to a group.

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(b) *If $m > 2$ then there is a group G, unique up to group isomorphism, such that Ω may be identified with G^m and the partitions in $\mathcal{Q}$ are the right-coset partitions of the subgroups $G_1, \ldots, G_m, \delta(G)$, where G_i has j-th entry 1 for all $j \neq i$, and $\delta(G)$ is the diagonal subgroup $\{(g, g, \ldots, g) : g \in G\}$.*

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- J. Dénes and A. D. Keedwell: *Latin Squares and their Applications*, Akadémiai Kaidó, Budapest, 1974.
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