Circular designs balanced for neighbours at distances one and two

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London Mathematical Society Regional Meeting,
Plymouth, 17 December 2014

Joint work with Tank Aldred (University of Otago, New Zealand), Brendan McKay (ANU, Australia) and Ian Wanless (Monash University, Australia)

Outline

- Some general considerations in the design and analysis of experiments where there may be an effect of neighbouring treatments.

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- The original problem, and first steps towards its solution.
- Two variants on the original problem, and their complete solution.
- What can we say about the original problem?

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Solution: Use a neighbour-balanced design in which each ordered pair (i, j) of different varieties occurs the same number of times as (South, North) neighbours.

Two designs for four varieties of sunflower

A	*B*	*C*	*D*
D	*A*	*B*	*C*
B	*C*	*D*	*A*
C	*D*	*A*	*B*
C	*D*	*A*	*B*

The Southern row consists of treated border plots on which no response is measured.

Each variety has each variety (including itself) just once as a Southern neighbour.

A	*B*	*C*	*D*
D	*A*	*B*	*C*
B	*C*	*D*	*A*
C	*D*	*A*	*B*
X	*X*	*X*	*X*

The Southern row is simply whatever is at the edge of the field.

Each variety has other each variety, and the field edge, just once as a Southern neighbour.

An experiment on control of aphids

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The aphids are as likely to spread East as West,
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Solution: Use a quasi-complete Latin square,
in which each unordered pair $\{i, j\}$ of sprays occurs
the same number of times as neighbours within rows
and the same number of times as neighbours within columns.

Five sprays on aphids

P	X	D	G	M
X	G	P	M	D
D	P	M	X	G
G	M	X	D	P
M	D	G	P	X

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P	X	D	G	M
X	G	P	M	D
D	P	M	X	G
G	M	X	D	P
M	D	G	P	X

Each pair of different treatments occurs
twice as row neighbours and twice as column neighbours.

Unequal replication (X denotes 'control')

X	P	D	M	G	X
M	X	P	G	X	D
D	G	M	P	X	X
G	D	X	X	M	P
X	M	X	D	P	G
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The experiment at Rothamsted on control of aphids

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The following year, many Rothamsted scientists used this design.
I had to beg them to come back to me to let me adapt it to their number of treatments.

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- Equal replication or not?

Data analysis: what do we want to do?

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  Do we want to test the hypothesis that all the brands of coffee taste the same (in which case we need power) or do we want to estimate how much better (on some scale) coffee A tastes than coffee B (in which case we want zero bias and small true variance)?
  Some people , in some applications, recommend using a neighbour-balanced design to reduce bias but fitting only the direct effects.

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  (for example, to make predictions about mixed plantings in future):
  now we need balance (or orthogonality) between different types of neighbour effects as well as between these and direct effects.
  In particular, if there are effects of neighbours from both the left and the right, then we need some sort of combinatorial balance at distance two even if the effects are felt only at distance one.

An experiment in marine biology

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A marine biologist wanted to compare 5 genotypes of bryozoan by suspending them in sea water around the circumference of a cylindrical tank. Each genotype was replicated 5 times, so that altogether 25 items were suspended in the tank.

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The marine biologist required that

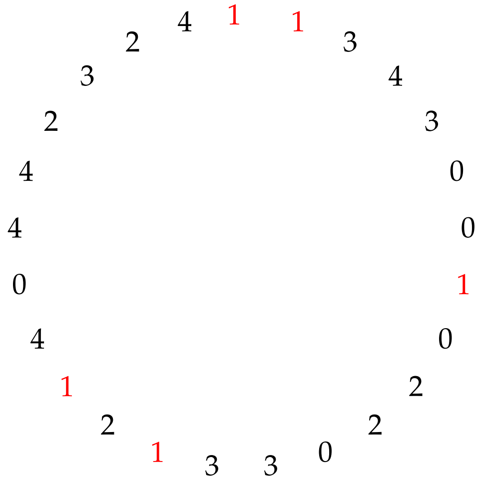
(i) each ordered pair of items should occur just once as ordered neighbours around the circumference of the tank;

(ii) each ordered pair of items should occur just once with a single item in between them, in order.

A circular design for 5 treatments with neighbour balance at distances one and two

1 1 3 4 3 0 0 1 0 2 2 0 3 3 1 2 1 4 0 4 4 2 3 2 4

A circular design for 5 treatments with neighbour balance at distances one and two



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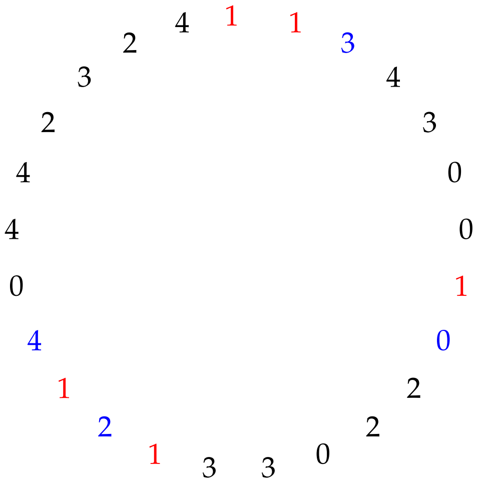

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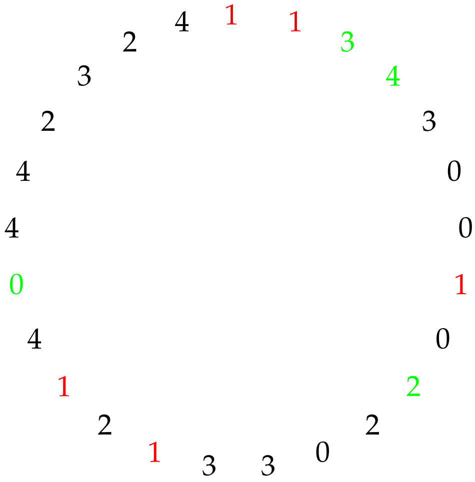
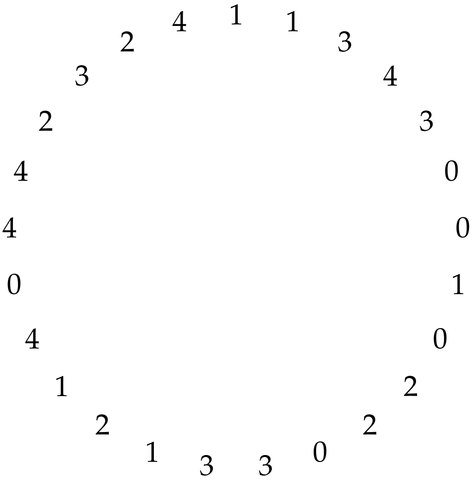


The lazy way to write the design

(1 1 3 4 3 0 0 1 0 2 2 0 3 3 1 2 1 4 0 4 4 2 3 2 4)



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The direct treatment effects δ,
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if and only if each pair (λ_j, δ_k) occurs equally often
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in other words, the design has neighbour balance at distances one and two.

Generalize the original problem

I wanted to prepare myself for future design requests like this.

Can we construct such a neighbour-balanced design
for n treatments each replicated n times
around a circle with space for n^2 items?

Those conditions again

Among the triples of the form

$$\big(\tau(i-1), \tau(i), \tau(i+1)\big),$$

each ordered pair of treatments occurs once in positions 1 and 2, once in positions 1 and 3, and once in positions 2 and 3.

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Among the triples of the form

$$\big(\text{row}, \text{column}, \text{letter}\big),$$

each ordered pair of symbols occurs once in positions 1 and 2, once in positions 1 and 3, and once in positions 2 and 3.

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These are conditions for a Latin square
whose rows and columns have the same labels as the letters
—a quasigroup.

Building the design from a quasigroup (Latin square)

The quasigroup operation $\circ$ is defined by

$$a \circ b = \text{letter in row } a \text{ and column } b \text{ of the Latin square.}$$

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We can start with any ordered pair (x, y) and successively build the circular design from the quasigroup as

$$x \quad y \quad x \circ y \quad y \circ (x \circ y) \quad (x \circ y) \circ (y \circ (x \circ y)) \quad \cdots$$

Latin square to circle

$\circ$	A	B	C	D
A	B	A	D	C
B	C	D	A	B
C	D	C	B	A
D	A	B	C	D

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$\circ$	A	B	C	D
A	B	A	D	C
B	C	D	A	B
C	D	C	B	A
D	A	B	C	D

$(\; A \quad A$

Latin square to circle

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A	B	A	D	C
B	C	D	A	B
C	D	C	B	A
D	A	B	C	D

$(\; A \quad A \quad B$

Latin square to circle

$\circ$	A	B	C	D
A	B	A	D	C
B	C	D	A	B
C	D	C	B	A
D	A	B	C	D

$(\; A \quad A \quad B$

Latin square to circle

$\circ$	A	B	C	D
A	B	A	D	C
B	C	D	A	B
C	D	C	B	A
D	A	B	C	D

$(\; A \quad A \quad B \quad A$

Latin square to circle

$\circ$	A	B	C	D
A	B	A	D	C
B	C	D	A	B
C	D	C	B	A
D	A	B	C	D

$(\; A \quad A \quad B \quad A$

Latin square to circle

$\circ$	A	B	C	D
A	B	A	D	C
B	C	D	A	B
C	D	C	B	A
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$$(A \quad A \quad B \quad A \quad C$$

Latin square to circle

$\circ$	A	B	C	D
A	B	A	D	C
B	C	D	A	B
C	D	C	B	A
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$(\; A \quad A \quad B \quad A \quad C$

Latin square to circle

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A	B	A	D	C
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A	B	A	D	C
B	C	D	A	B
C	D	C	B	A
D	A	B	C	D

$$(\; A \quad A \quad B \quad A \quad C \quad D \quad A$$

Latin square to circle

$\circ$	A	B	C	D
A	B	A	D	C
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A	B	A	D	C
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$(\; A \quad A \quad B \quad A \quad C \quad D \quad A \quad A$ oops!

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This quasigroup gives a design with four separate circles, not one.

$$(\; A \quad A \quad B \quad A \quad C \quad D \;)$$

$$(\; A \quad D \quad C \quad C \quad B \quad C \;)$$

$$(\; B \quad B \quad D \;)$$

$$(\; D \;)$$

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	0	1	2	3	4
0	1	0	2	3	4
1	2	3	1	4	0
2	3	4	0	2	1
3	0	2	4	1	3
4	4	1	3	0	2

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$$(1\ 1\ 3\ 4\ 3\ 0\ 0\ 1\ 0\ 2\ 2\ 0\ 3\ 3\ 1\ 2\ 1\ 4\ 0\ 4\ 4\ 2\ 3\ 2\ 4)$$

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BUT we have been unable to prove that they always exist.

It is quite easy to show that, if $Q = \mathbb{Z}_{p^s}$ or $Q = \mathrm{GF}(p^s)$,
then no binary operation of the form

$$x \circ y = ax + by + c$$

makes Q into an Eulerian quasigroup.

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In September 2004 I spent two weeks at ANU working with BDM and IMW (and remotely with RELA). We solved the two variants completely.

Variant I: no self-neighbours

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We need a circular design with $n(n-1)$ plots in which each

- each ordered pair of distinct treatments occurs just once as ordered neighbours;
- each left-neighbour treatment occurs just once with all but one of the right-neighbour treatments.

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The incidence of
direct treatments with left-neighbour treatments is a symmetric balanced incomplete-block design (BIBD, aka 2-design);

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We need a circular design with $n(n-1)$ plots in which each

- each ordered pair of distinct treatments occurs just once as ordered neighbours;
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The incidence of
direct treatments with left-neighbour treatments is a symmetric balanced incomplete-block design (BIBD, aka 2-design);
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Preece (1976) showed that, for overall balance, the missing pairs at distance two must also be the self-pairs.

Idempotent Eulerian circular sequences

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Our circular design is equivalent to an idempotent quasigroup in which the $n(n-1)$ off-diagonal cells give a single circle.

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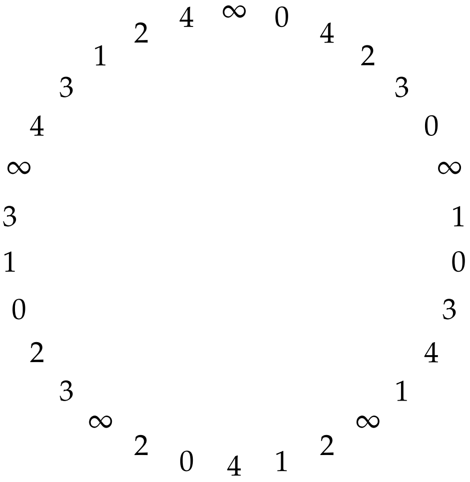

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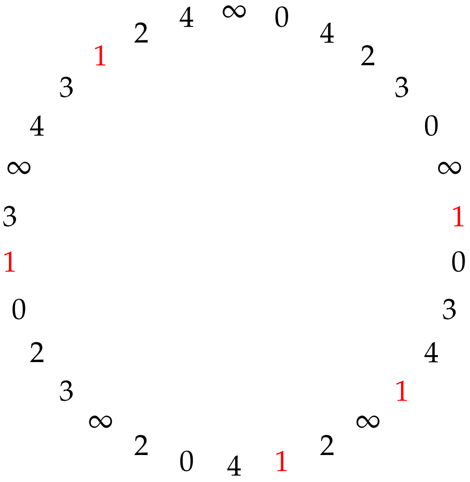

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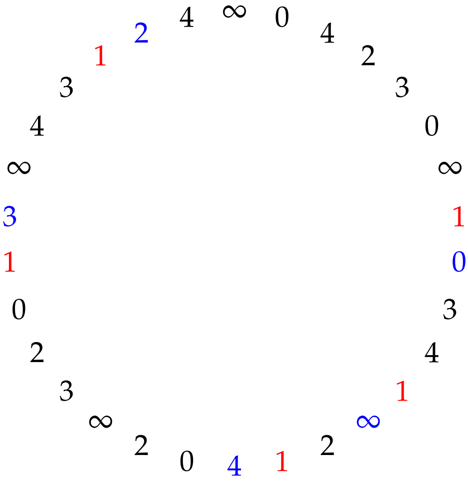

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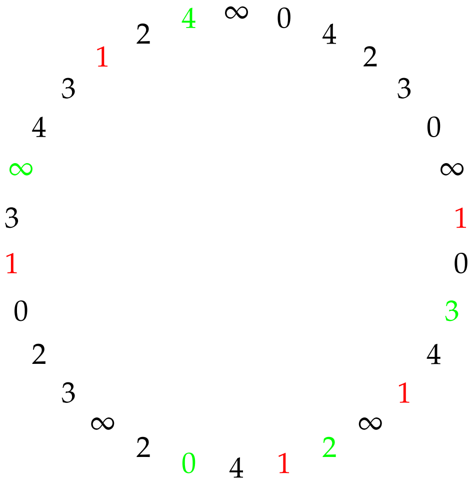

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Solution for variant I

Theorem

Given an initial sequence of the non-zero integers modulo $n-1$ satisfying those conditions,
that construction always produces an idempotent Eulerian circular sequence.

Theorem

Such an initial sequence can be constructed whenever $n \geq 6$.

Paradigm

$$\text{Small thing} \xrightarrow{\qquad\qquad\text{algorithm}\qquad\qquad} \text{Large thing}$$

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- Prove that it works.

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Any triple (a, b, a) gives b as a neighbour of a on both sides, so there can be no such triples.

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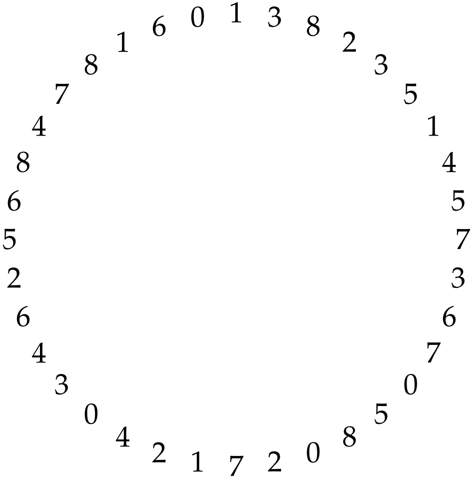
Differences at distance one come from the original sequence; difference at distance two are the neighbour sums.

A circular design for 9 treatments with undirectional neighbour balance at distances one and two



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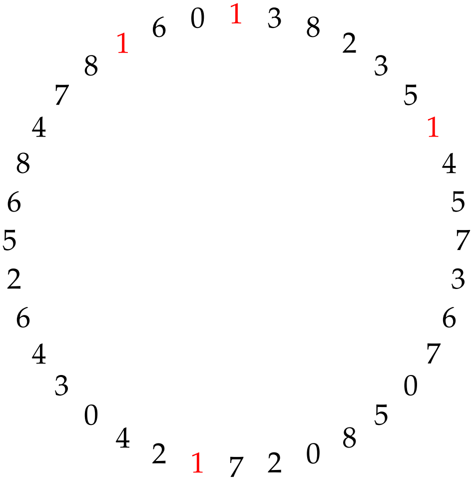

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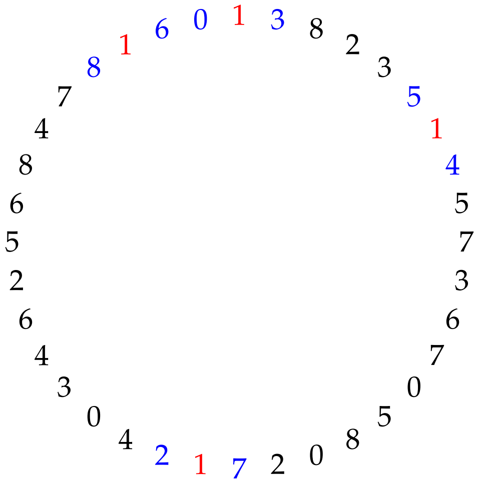

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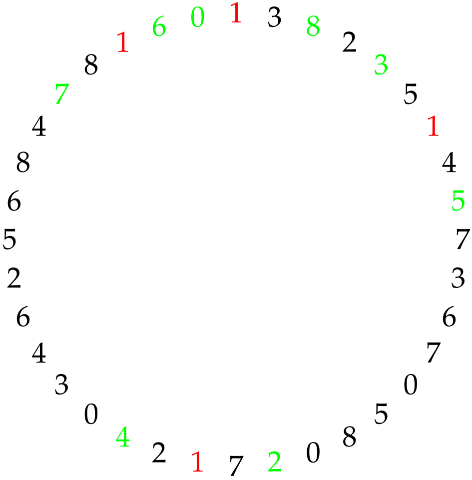

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Solution for variant II

Theorem

Given an initial circular sequence of $(n-1)/2$ *of the integers modulo n satisfying those conditions, that construction always produces a circular sequence balanced for undirected neighbours at distances one and two.*

Theorem

Such an initial sequence can be constructed whenever n is odd and $n \geq 9$*. There is also such a circular sequence when* $n = 7$*.*

Back to the original question

A quasigroup of order n with operation $\circ$ is Eulerian if the sequence

$$x \quad y \quad x \circ y \quad y \circ (x \circ y) \quad (x \circ y) \circ (y \circ (x \circ y)) \quad \cdots$$

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Conjecture

If $n \geq 5$ then there exists an Eulerian quasigroup of order n.

Coprime sizes

Theorem

If $(Q_1, \bullet)$ and $(Q_2, \circ)$ are Eulerian quasigroups of orders n and m, where n and m are coprime, then $Q_1 \otimes Q_2$ is an Eulerian quasigroup of order nm.

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where n *and* m *are coprime,*
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Proof.

In the sequence

$$(a, x) \quad (b, y) \quad (a \bullet b, x \circ y) \quad (b \bullet (a \bullet b), y \circ (x \circ y)) \quad \cdots$$

the first coordinates repeat every n^2 steps, but not earlier, and the second coordinates repeat every m^2 steps, but not earlier. □

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> *Back in Australia now and awake in the middle of the night... but wanted to let you know that in my sleeplessness I've solved that parity question.*

We still have no general construction, but a paper eventually got written and submitted.

Strategy

So all we have to do is to find an Eulerian quasigroup for all of the following orders:

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(and the paper had been accepted before we realised that we also need)

- $3 \times$ all non-trivial powers of 2.

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If p is prime and $Q = \mathbb{Z}_p$, then no binary operation of the form

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If $a + b - 1 = 0$ and $b \neq 2$ and $t = -(b - 2)^{-1}c$
then $mt \circ (m + 1)t = (m + 2)t$ for all integers m,
so we get a circle of size p.

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If $a + b - 1 = 0$ and $b = 2$
then ${}^{m}\mathrm{C}_2 c \circ {}^{m+1}\mathrm{C}_2 c = {}^{m+2}\mathrm{C}_2 c$ for all positive integers m,
so we get a circle of size p.

Technique to avoid brute search

If q is odd, try taking $Q = \mathbb{Z}_q$ and putting

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For example, when $q = 7$ put $\pi = (0\,1\,2)(3\,4)$ so that

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This (the permutation $(0\ 1\ 2)$ with some adjacent transpositions) works for all odd numbers that we have tried.

That parity obstacle

Theorem

If n is even then no Eulerian quasigroup can be obtained from a group of order n by permutions of rows, columns or symbols.

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Theorem

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… so IMW found another technique to cut down the computer search when n is even.

. . . for all practical purposes

Theorem
If $n \geq 5$ and there is no Eulerian quasigroup of order n then n is divisible by a prime power exceeding 1000.

References I

- Aldred, R. E. L., Bailey, R. A., McKay, B. D. and Wanless, I. M.:
  Circular designs balanced for neighbours at distances one and two.
  Biometrika, **101** (2014), 943–956.
- Azaïs, J.-M., Bailey, R. A. and Monod, H.:
  A catalogue of efficient neighbour-designs with border plots.
  Biometrics, **49** (1993), 1252–1261.
- Bailey, R. A.:
  Quasi-complete Latin squares: construction and randomization.
  Journal of the Royal Statistical Society, Series B **46** (1984), 323–334.

- Bailey, R. A. and Druilhet, P.:
  Optimality of neighbour-balanced designs for total effects.
  Annals of Statistics **32** (2004), 1650–1661.
- Bailey, R. A. and Druilhet, P.:
  Optimal cross-over designs for full interaction models.
  Annals of Statistics, **42** (2014), 2282–2300.
- Bayer, M. M. and Todd, C. D.:
  Effect of polypide regression and other parameters on colony growth in the cheilostomate Electra pilosa (L.).
  In *Bryozoans in Space and Time* (eds. D. P. Gordon, A. M. Smith and J. A. Grant-Mackie), pp. 29–38.
  Wellington, NZ: National Institute of Water and Atmospheric Research (1996).

References III

- David, O., Monod, H., Lorgeau, J. and Philippeau, G.:
  Control of interplot interference in grain maize: a multi-site comparison.
  Crop Science, **41** (2001), 406–414.
- Druilhet, P.:
  Optimality of neighbour-balanced designs.
  Journal of Statistical Planning and Inference, **81** (1999), 141–152.
- Preece, D. A.:
  Non-orthogonal Graeco-Latin designs.
  In *Combinatorial Mathematics IV* (eds. L. R. A. Casse and W. D. Wallis), Lecture Notes in Mathematics, **560**, pp. 7–26. Berlin: Springer (1976).