Resistance distance in the context of association schemes and coherent configurations

R. A. Bailey
University of St Andrews

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Joint work with Peter Cameron (University of St Andrews), and Michael Kagan (Penn State University)

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The corresponding $\Omega \times \Omega$ square matrices (with all entries equal to 1 or 0) are I, A_Γ and $J - A_\Gamma - I$, where I is the identity matrix, A_Γ is the adjacency matrix of Γ, and J is the all-1 matrix.

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Let $\mathcal{A}_\Gamma$ be the set of real linear combinations of these matrices. Then $\mathcal{A}_\Gamma$ is closed under matrix multiplication if and only if the graph Γ is strongly regular.

Generalize to other partitions of $\Omega \times \Omega$

Let W be any subset of $\Omega \times \Omega$. We generalize the idea of adjacency matrix by writing A_W for the $\Omega \times \Omega$ matrix with

$$A_W(\alpha, \beta) = \begin{cases} 1 & \text{if } (\alpha, \beta) \in W \\ 0 & \text{otherwise.} \end{cases}$$

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We are going to consider three conditions (and their variants) that Π might satisfy.

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The partial order on partitions of $\Omega \times \Omega$

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The supremum, or join, of partitions Ψ_1 and Ψ_2 is the partition $\Psi_1 \vee \Psi_2$ which satisfies $\Psi_1 \preccurlyeq \Psi_1 \vee \Psi_2$ and $\Psi_2 \preccurlyeq \Psi_1 \vee \Psi_2$ and if $\Psi_1 \preccurlyeq \Phi$ and $\Psi_2 \preccurlyeq \Phi$ then $\Psi_1 \vee \Psi_2 \preccurlyeq \Phi$.

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Draw a graph by putting an edge between two points if they are in the same part of Ψ_1 or the same part of Ψ_2. Then the parts of $\Psi_1 \vee \Psi_2$ are the connected components of the graph.

Association schemes and coherent configurations

Suppose that Φ and Ψ are both association schemes.
Then $\Phi \vee \Psi$ is also an association scheme.
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Let Π be any partition of $\Omega \times \Omega$. Then the set of coherent configurations which refine Π is non-empty. The supremum of all of these is a coherent configuration $\mathrm{CC}(\Pi)$ satisfying

1. $\mathrm{CC}(\Pi) \preccurlyeq \Pi$;
2. if Φ is a coherent configuration then $\Phi \preccurlyeq \Pi$ if and only if $\Phi \preccurlyeq \mathrm{CC}(\Pi)$.

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There are n paths of length two from α to γ (including (α, α, γ) and (α, γ, γ)). If we re-label the pair (α, γ) according to how many such pairs have each ordered pair of colours, then we obtain a new partition of $\Omega \times \Omega$.

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There are n paths of length two from α to γ (including (α, α, γ) and (α, γ, γ)). If we re-label the pair (α, γ) according to how many such pairs have each ordered pair of colours, then we obtain a new partition of $\Omega \times \Omega$.

We shall call this $\mathrm{WL}(\Pi)$, because this uses the algorithm introduced by Weisfeiler and Leman. It is clear that $\mathrm{WL}(\Pi) \preccurlyeq \Pi$.

Properties of $\mathrm{WL}(\Pi)$

Suppose that Π satsifies (C1).
If red is a colour then it has a dual colour red′ (which may be the same as red) such that $A_{\mathrm{red}'} = A_{\mathrm{red}}^{\top}$. The reverse of a path coloured $(\mathrm{red}, \mathrm{blue})$ from α to β is a path coloured $(\mathrm{blue}', \mathrm{red}')$ from β to α. Hence $\mathrm{WL}(\Pi)$ also satisfies (C1).

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Suppose that Π satisfies (C2).
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In general, the number of $(\mathrm{red}, \mathrm{blue})$ paths from α to β is the (α, β)-entry in $A_{\mathrm{red}}A_{\mathrm{blue}}$. Thus if Π satisfies (C3) then $\mathrm{WL}(\Pi) = \Pi$. Otherwise, $\mathrm{WL}(\Pi) \prec \Pi$.

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These results show that if Π is a coherent configuration then $\Pi = \mathrm{WL}(\Pi)$. On the other hand, if Π satisfies (C1) and (C2) but not (C3) then WL(Π) satisfies (C1) and (C2) and $\mathrm{WL}(\Pi) \prec \Pi$.

Applying Weisfeiler–Leman to a graph

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Moreover, it is easy to see that if $\Pi_1 \succcurlyeq \Pi_2$ then $\mathrm{WL}(\Pi_1) \succcurlyeq \mathrm{WL}(\Pi_2)$. Therefore,

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Each time that Weisfeiler–Leman is applied, either the resulting partition is strictly finer than the preceding one or the preceding one is $\mathrm{CC}(\Pi_\Gamma)$. Since Π_Γ has finitely many classes, the process stabilizes at $\mathrm{CC}(\Pi_\Gamma)$ after finitely many steps.

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Michael Kagan and Misha Klin proposed an alternative method using resistance distance, which I will now explain.

Electrical networks

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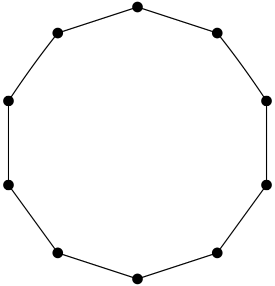
Find the total current I from α to β, then use Ohm's Law to define the resistance distance $R_{\alpha\beta}$ between α and β as $1/I$.

Resistance distance in two sparse graphs with 10 vertices

The cycle.



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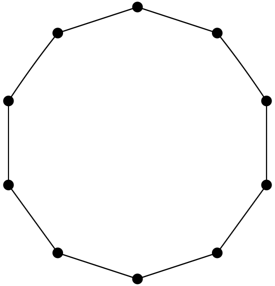

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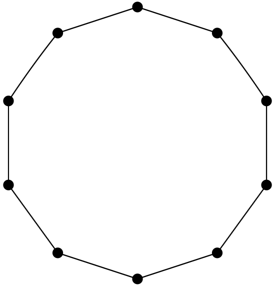

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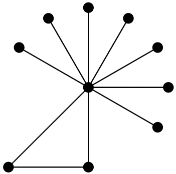

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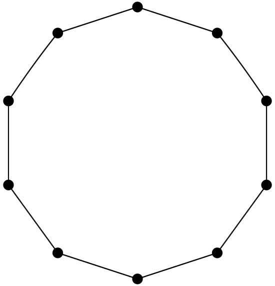

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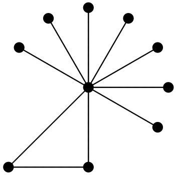

$R_{\alpha\beta} \leq 2$ for all α, β.

Using the Laplacian matrix

Definition

The Laplacian matrix L of the graph Γ is the $\Omega \times \Omega$ matrix with

$$L_{\alpha\beta} = \begin{cases} \text{degree of } \alpha & \text{if } \alpha = \beta \\ -1 & \text{if } (\alpha, \beta) \text{ is an edge} \\ 0 & \text{otherwise.} \end{cases}$$

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$$M = \left(L + \frac{1}{n}J\right)^{-1} - \frac{1}{n}J$$

and M is a polynomial in L.

Theorem

The resistance distance $R_{\alpha\beta}$ between vertices α and β is

$$R_{\alpha\beta} = \left(M_{\alpha\alpha} + M_{\beta\beta} - 2M_{\alpha\beta}\right).$$

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Then RDT can be iterated.

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Does iterated RDT always produce a symmetrized version of iterated WL?

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MKMK talked about RDT at a conference in Pilsen in 2018 and one in Yichang in 2019.

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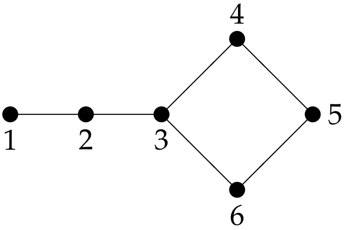
We decided to join forces.

Some small examples: Bad results



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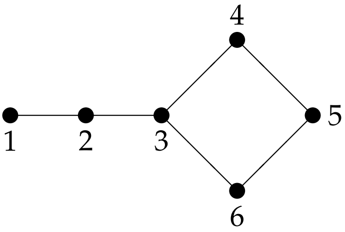

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

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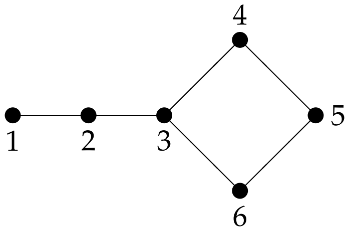

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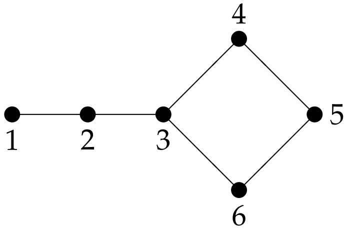
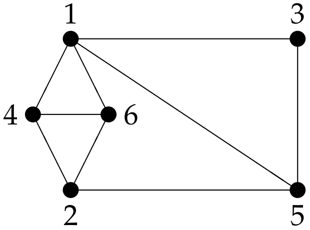
$\{2,3\}$ is an edge; $\{4,6\}$ is a non-edge; but $R_{23} = R_{46} = 1$.
In this case, the partition defined by resistance distance does not refine the original partition Π_Γ.

Bad results for the complementary graph





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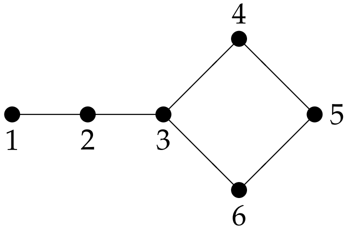

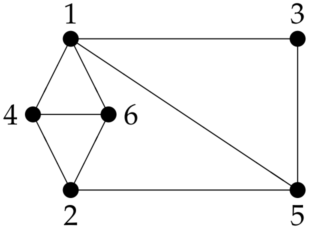

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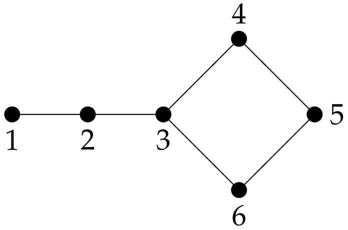

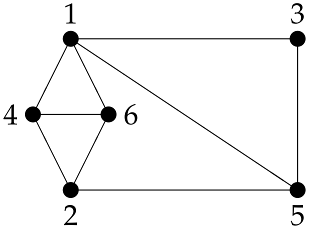

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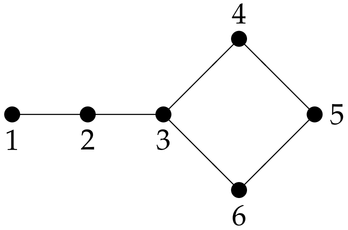

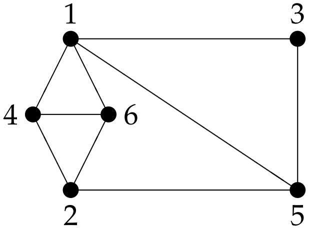

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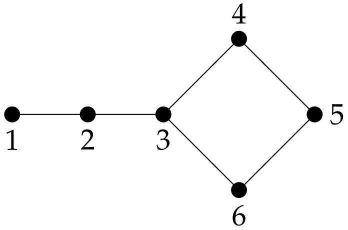

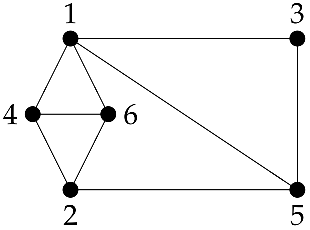

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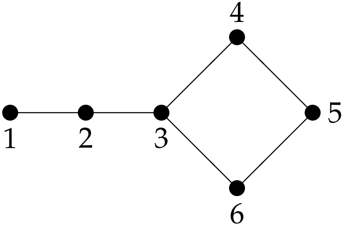

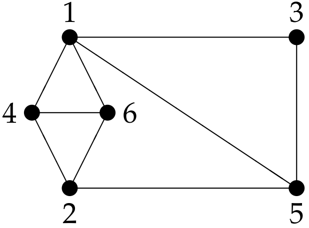

$$\begin{bmatrix} 0 & 1 & 2 & \frac{11}{4} & 3 & \frac{11}{4} \\ 1 & 0 & 1 & \frac{7}{4} & 2 & \frac{7}{4} \\ 2 & 1 & 0 & \frac{3}{4} & 1 & \frac{3}{4} \\ \frac{11}{4} & \frac{7}{4} & \frac{3}{4} & 0 & \frac{3}{4} & 1 \\ 3 & 2 & 1 & \frac{3}{4} & 0 & \frac{3}{4} \\ \frac{11}{4} & \frac{7}{4} & \frac{3}{4} & 1 & \frac{3}{4} & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & \frac{5}{8} & \frac{5}{8} & \frac{17}{32} & \frac{1}{2} & \frac{17}{32} \\ \frac{5}{8} & 0 & 1 & \frac{17}{32} & \frac{5}{8} & \frac{17}{32} \\ \frac{5}{8} & 1 & 0 & \frac{33}{32} & \frac{5}{8} & \frac{33}{32} \\ \frac{17}{32} & \frac{17}{32} & \frac{33}{32} & 0 & \frac{25}{32} & \frac{1}{2} \\ \frac{1}{2} & \frac{5}{8} & \frac{5}{8} & \frac{25}{32} & 0 & \frac{25}{32} \\ \frac{17}{32} & \frac{17}{32} & \frac{33}{32} & \frac{1}{2} & \frac{25}{32} & 0 \end{bmatrix}$$

In the complementary graph Γ', for resistance distances not involving 4 or 6, we can replace the left-hand side by a single edge between 1 and 2. Therefore $R'_{12} = R'_{13}$. But $R_{12} \neq R_{13}$. Neither of these RDT partitions refines the other.

More sophisticated resistance distance transform

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We have not found an example where edges with different indeterminates get the same resistance distance, but we have not yet proved that this does not happen.

Some positive results

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If Π *is an association scheme, then* $\mathrm{RDT}\,2(\Pi) = \Pi$.

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Theorem

If Γ *is the graph corresponding to a non-diagonal part in an association scheme, and RDT2 is applied repeatedly, then it stabilizes at the association scheme which is the supremum of all association schemes which have* Γ *as such a graph.*

Some questions

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2. If so, does it do so in fewer steps than Weisfeiler–Leman?
3. If $CC(\Pi)$ does not satisfy (C1+), does RDT2 stabilize at a Jordan scheme?