

Weakly neighbour-balanced designs

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Joint work with Katarzyna Filipiak and Augustyn Markiewicz (Poznan University of Life Sciences), Joachim Kunert (TU Dortmund) and Peter Cameron (St Andrews)

Small example: each treatment comes “once” per block

Wind →

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1:0	6	5	4	3	2	1

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RAB and PJC gave some constructions and non-existence results.

A 0, 1-matrix

If we have a design which is weakly neighbour balanced but not neighbour balanced then S has zero diagonal, some other entries $\lambda - 1$ and some other entries λ . Put

$$A = S - (\lambda - 1)(J - I).$$

Then

- ▶ A is not zero;
- ▶ all entries of A are in $\{0, 1\}$;
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We know something about (some) matrices like this!

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Theorem

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Number the positions in each block $1, 2, \dots$, starting at the windy end.

Theorem

If a WNBD has the property that each numbered position has all treatments equally often, then it either is a NBD or has Type I.

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$$0 \neq y^2 \in Z_t \quad \begin{array}{cccc} & x \in Z_t & x + 1 & \\ & & & \\ & & & \\ \dots & xy^2 & (x + 1)y^2 & \dots \\ & & & \\ & & & \end{array} \quad \text{is a WNBD.}$$

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$t = 15$? RAB tried using A as the incidence matrix of $\text{PG}(3, 2)$ and proved that it is impossible.

Type I and $t = 15$

Reid and Brown give the following doubling construction.

$$A_2 = \begin{pmatrix} A_1^\top & 0_t & A_1 + I_t \\ \mathbf{1}_t^\top & 0 & \mathbf{0}_t^\top \\ A_1 & \mathbf{1}_t & A_1 \end{pmatrix}$$

If A_1 is Type I for t then A_2 is Type I for $2t + 1$.

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Doing this with $t = 7$ gives a doubly regular tournament Γ_2 on 15 vertices with an automorphism π of order 7.

If we can find a Hamiltonian cycle φ in Γ_2 which has no edge in common with any of $\pi^i(\varphi)$ for $i = 1, \dots, 6$, then $\varphi, \pi(\varphi), \dots, \pi^6(\varphi)$ make a WNBD.

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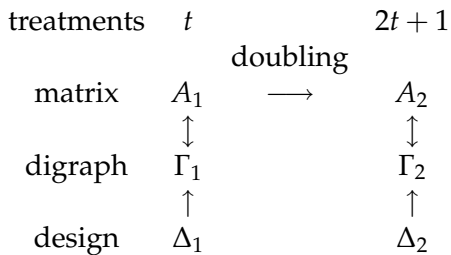
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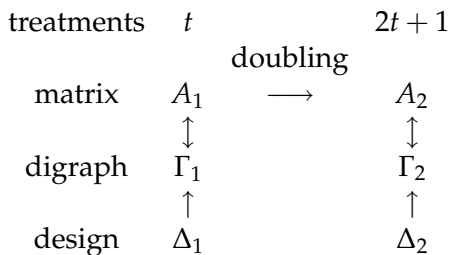
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KF put this A_2 into Mathematica and asked it to find Hamiltonian decompositions.

Question



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Could we go directly from Δ_1 to Δ_2 ?

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We can also take advantage of symmetry to find a single Hamiltonian cycle whose images under a group of automorphisms of Γ give the blocks of the WNBD.

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Again, is there a way of going directly from the smaller design to the larger one?