# Weakly neighbour-balanced designs

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Joint work with Katarzyna Filipiak and Augustyn Markiewicz (Poznan University of Life Sciences), Joachim Kunert (TU Dortmund) and Peter Cameron (St Andrews)

# Small example: each treatment comes “once” per block

Wind →

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 6 ⋮ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 5 ⋮ | 0 | 2 | 4 | 6 | 1 | 3 | 5 |
| 3 ⋮ | 0 | 4 | 1 | 5 | 2 | 6 | 3 |
| 6 ⋮ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 5 ⋮ | 0 | 2 | 4 | 6 | 1 | 3 | 5 |
| 4 ⋮ | 0 | 3 | 6 | 2 | 5 | 1 | 4 |
| 3 ⋮ | 0 | 4 | 1 | 5 | 2 | 6 | 3 |
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RAB and PJC gave some constructions and non-existence results.

# A 0, 1-matrix

If we have a design which is weakly neighbour balanced but not neighbour balanced then $S$ has zero diagonal, some other entries $\lambda - 1$ and some other entries $\lambda$. Put

$$A = S - (\lambda - 1)(J - I).$$

Then

- $A$ is not zero;
- all entries of $A$ are in $\{0, 1\}$;
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We know something about (some) matrices like this!

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If Type II, then $A^\top A$ is completely symmetric.
If Type III, then $A^\top A$ is not completely symmetric.

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Theorem

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Number the positions in each block $1, 2, \ldots,$ starting at the windy end.

**Theorem**

*If a WNBD has the property that each numbered position has all treatments equally often, then it either is a NBD or has Type I.*

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|---|---|---|---|---|
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$t = 7$ ✓

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$t = 15$? RAB tried using $A$ as the incidence matrix of $\mathrm{PG}(3, 2)$ and proved that it is impossible.

# Type I and $t = 15$

Reid and Brown give the following doubling construction.

$$A_2 = \begin{pmatrix} A_1^\top & 0_t & A_1 + I_t \\ 1_t^\top & 0 & 0_t^\top \\ A_1 & 1_t & A_1 \end{pmatrix}$$

If $A_1$ is Type I for $t$ then $A_2$ is Type I for $2t+1$.

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Doing this with $t = 7$ gives a doubly regular tournament $\Gamma_2$ on 15 vertices with an automorphism $\pi$ of order 7.
If we can find a Hamiltonian cycle $\varphi$ in $\Gamma_2$ which has no edge in common with any of $\pi^i(\varphi)$ for $i = 1, \ldots, 6$, then $\varphi, \pi(\varphi), \ldots, \pi^6(\varphi)$ make a WNBD.

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PJC used GAP, and found 120 solutions.
KF put this $A_2$ into Mathematica and asked it to find Hamiltonian decompositions.

# Question

$$
\begin{array}{cccc}
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\text{matrix} & A_1 & \longrightarrow & A_2 \\
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Could we go directly from $\Delta_1$ to $\Delta_2$?

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Using familiar tricks for constructing BIBDs (such as perfect difference sets), we can construct WNBDs.

We can also take advantage of symmetry to find a single Hamiltonian cycle whose images under a group of automorphisms of $\Gamma$ give the blocks of the WNBD.

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and $\begin{pmatrix} 0 & 1_t^\top & 0 & 0_t^\top \\ 0_t & A_1 & 1_t & A_1 \\ 0 & 0_t^\top & 0 & 1_t^\top \\ 1_t & A_1^\top & 0_t & A_1 \end{pmatrix}$ has Type III for $2(t+1)$ treatments.

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and $$\begin{pmatrix} 0 & 1_t^\top & 0 & 0_t^\top \\ 0_t & A_1 & 1_t & A_1 \\ 0 & 0_t^\top & 0 & 1_t^\top \\ 1_t & A_1^\top & 0_t & A_1 \end{pmatrix}$$ has Type III for $2(t+1)$ treatments.

$t = 3$ leads to the only Type III WNBDs ($t = 6$ and $t = 8$) found by KF and AM.

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Some S-digraphs (Babai and Cameron) satisfy this.
If $A_1$ has Type I for $t$ treatments then

$$\begin{pmatrix} A_1 & A_1+I_t & \dots & A_1+I_t \\ A_1+I_t & A_1 & \dots & A_1+I_t \\ \vdots & \vdots & \ddots & \vdots \\ A_1+I_t & A_1+I_t & \dots & A_1 \end{pmatrix} \quad \text{has Type III for } mt \text{ treatments}$$

and $\begin{pmatrix} 0 & 1_t^\top & 0 & 0_t^\top \\ 0_t & A_1 & 1_t & A_1 \\ 0 & 0_t^\top & 0 & 1_t^\top \\ 1_t & A_1^\top & 0_t & A_1 \end{pmatrix}$ has Type III for $2(t+1)$ treatments.

$t=3$ leads to the only Type III WNBDs ($t=6$ and $t=8$) found by KF and AM.
Again, is there a way of going directly from the smaller design to the larger one?